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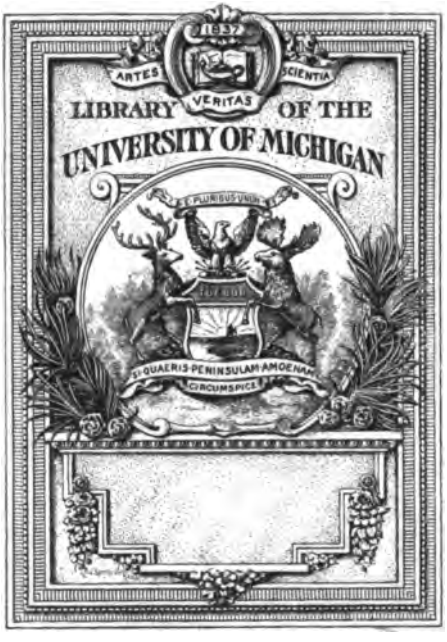

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The Editors of the Mathematician, on completing the second volume of that work, beg leave to express their most cordial thanks to their friends and supporters.

In soliciting a renewal of their liberal support for the future, the Editors can with much confidence appeal to the past, and they feel sure that their exertions in the cause of science will continue to secure the co-operation and assistance of the friends of science in general.

ROYAL MILITARY ACADEMY,
July 1, 1847.

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INDEX TO VOL. II.

Algebra.

PAGE

On the Summation of Infinite Series..... 6

Computation of Logarithms and Anti-Logarithms 17, 249

Horner on Algebraic Transformation 32, 129

On the Peculiar Equation $x^3 - bx = 1$..... 253, 319

Algebraic Demonstration of the Theorem

$$1.2.....n = n^n - n(n-n)^n + \frac{n(n-1)}{1.2}(n-2)^n..etc.... 296$$

Plane Geometry.

Application of Algebra to the Modern Geometry 1, 84, 143, 237

New Demonstration of the Theorems of Pascal and Brianchon..... 15

On Confocal Conic Sections 26

Demonstration of a Geometrical Theorem..... 37

Propositions on the Conic Sections.....61, 132

On Involution..... 63

Relations between the Radii of certain Circles..... .. 69

Note on a New Geometrical Theorem 74

On the Tangents to Algebraic Curves..... 83

Note on the Tangent to the Parabola 96

Properties of Conic Sections and their inscribed and circumscribed Parallelograms 119

On the Theory of Collineation.....123, 189,

Properties of the Ellipse..... 179

On the greatest inscribed and least circumscribed Ellipse to a given Quadrilateral Figure..... 229

Properties of the inscribed and circumscribed Quadrilaterals to a System of Curves of the Second Order..... 285

Properties of the Centres of Gravity of a Quadrilateral and the Triangles of which it is composed..... 292

On the minimum Ellipse passing through four given Points..... 315

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Solid Geometry.

PAGE

A Property relative to Surfaces which pass through the same Eight Points in Space 12
On the Theorems in Space analogous to a Certain Property of Curves of the Second Order 66
On Surfaces of the Second Order 80
The Modern Geometry 91, 199, 310
On Propositions in Plano and their analogous Properties in Space 117
On the Circular Sections of Surfaces of the Second Order 173
Surfaces of the Second Order referred to Three Tangent Planes, *etc.* 177, 258
On Tangent Planes 187
On the Determination of the Nature of Surfaces of the Second Order 257
Division of the Ellipsoid into Equal Volumes 302

Plane and Spherical Trigonometry.

Direct Solution of a Spherical Problem 31
Simplifications in Elementary Trigonometry 68
Connection between the Formulæ in Plane and Spherical Trigonometry 75
Computation of Trigonometrical Tangents and Anti-tangents 135
Elementary Investigation of the Formula for computing the Equation of Equal Altitudes 148

Mechanics.

On the Vertical Force Magnetometer 149
Note on Pratt's Demonstration of the Parallelogram of Forces 249
The Coordinates of the Centre of Gravity of a Figure bounded by Straight Lines or Planes 254

Differential Equations.

On the General Reduction of Certain Integrals 28, 75, 87
On a New Method of Integrating Linear Differential Equations 204
On Certain Definite Integrals expressible by means of Elliptic Functions 297

Miscellaneous.

Mathematical Notes 16, 32, 39, 134, 235, 320, 321
Mathematical Exercises 40, 115, 171, 227, 283, 339
Solutions of Mathematical Exercises 41, 97, 150, 209, 263, 321
Note by the Editors relative to their Elementary Propositions in the Geometry of Coordinates 295

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THE MATHEMATICIAN.

VOL. II. NOVEMBER, 1845. No. I.

APPLICATION OF ALGEBRA TO THE MODERN GEOMETRY.

[*Mr. Finlay, Professor of Mathematics and Natural Philosophy, Manchester New College.*]

(Continued from vol. i. p. 397.)

SECTION II.

ON THE INTERSECTION AND CONTACT OF SURFACES.

Having given some simple illustrations of the methods employed in this paper by applying them to the geometry of plane curves, I shall now proceed to apply them to surfaces. For the sake of distinctness this section is divided into three parts. The first contains investigations of the principal properties of plane sections of surfaces of the second degree. The second traces the connexion of these properties with the relations of involution that exist among the points in which an arbitrary transversal meets any system of surfaces of the first and second degree. And the third contains extensions of these theorems to surfaces of the higher degrees. With two or three exceptions, I believe the theorems at which I have arrived to be entirely new.

Plane sections of surfaces of the second degree.

I.

To find the equation of a surface of the second degree passing through two given plane sections of a given surface of the second degree.

Let the given surface be denoted by

$$u = Ax^2+By^2+Cz^2+Dyz+Ezx+Fxy+Gx+Hy+Kz+1=0 \ldots (1)$$

and the given planes which intersect it by

$$v = ax+\beta y+\gamma z+1=0 \ldots (2)$$

$$v' = a'x+\beta' y+\gamma' z+1=0 \ldots (3)$$

then will the required surface be represented by

$$u = mvv' \ldots (4)$$

m being an arbitrary constant. For, by subtracting (1) from (4) we get $mvv'=0$; which shows that the intersection (real or imaginary) of the surfaces (1) and (4) lies upon the planes (2) and (3).

Since m is arbitrary, it is evident that *an infinite number of surfaces of the second degree may be described so as to pass through any two plane sections of a given surface of the second degree.*

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II.

If a surface of the second degree intersect a given surface of the second degree along a given plane curve, and pass through two given points not on the given surface; the other plane in which it intersects the given surface* passes through a fixed point, which lies on the straight line connecting the two given points.

If $u = 0$ denote the equation of the given surface, $v' = 0$ that of the given plane, $v = 0$ the equation of the other plane section; the equation of the variable surface will be (1)

$$u = mvv'.$$

If we put $\frac{u}{v'} = t$, this becomes $mv = t$; or

$$ax + by + cz + m = t \ldots\ldots\ldots\ldots\ldots\ldots (4')$$

where $a = m\alpha$, $b = m\beta$, $c = m\gamma$. Let $x'y'z'$, $x''y''z''$ be the two given points, and let t' be the value of t when $x'y'z'$ are put for xyz: then since the surface passes through these points, we shall have

$$ax' + by' + cz' + m = t' \ldots\ldots\ldots\ldots\ldots\ldots (a)$$
$$ax'' + by'' + cz'' + m = t'' \ldots\ldots\ldots\ldots\ldots\ldots (b)$$

and by multiplying the former by t'', the latter by t', and taking the difference of the results, we get

$$a(x't'' - t'x'') + b(y't'' - t'y'') + c(z't'' - t'z'') + m(t'' - t') = 0;$$

which shows that the plane $v = 0$, or

$$ax + by + cz + m = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2')$$

passes through the fixed point whose co-ordinates are

$$\frac{x't'' - x''t'}{t'' - t'}, \frac{y't'' - y''t'}{t'' - t'}, \frac{z't'' - z''t'}{t'' - t'} \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The equations of the straight line joining the two given points being

$$x - x' = \frac{x' - x''}{z' - z''}(z - z'),\ y - y' = \frac{y' - y''}{z' - z''}(z - z');$$

are satisfied by substituting the expressions (5) for x, y, z; and therefore the point (5) lies on the straight line joining the given points. Hence we have the following theorem:

If a surface of the second degree intersect a given surface of the second degree along a given plane curve, and pass through two given points not on the given surface; and if the surface vary gradually in magnitude, the other plane along which it intersects the given surface will turn about a fixed point in the straight line joining the two given points.

When the given points lie on the given surface $u = 0$, but not on its intersection with the plane $v = 0$, we have $t' = 0$ and $t'' = 0$; and equations (a) and (b) become

$$ax' + by' + cz' + m = 0,$$
$$ax'' + by'' + cz'' + m = 0;$$

hence, in this case, the plane (2') passes through the two given points, and we have the following theorem:

* We assume here that if two surfaces of the second degree intersect along a plane curve, they will also intersect, in general, along a second plane curve. The demonstration of this proposition is given in the ordinary elementary books.

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If a surface of the second degree intersect a given surface of the second degree in a given plane curve, and pass through two given points on the given surface, and if the surface vary in magnitude: the other plane along which it intersects the given surface will turn about the straight line joining the two given points.

III.

If a surface of the second degree cut a given surface of the second degree in a given plane curve, and pass through three given points not on the given surface; the other plane in which it intersects the given surface passes through three fixed points, which lie in the same straight line.

Let $x'y'z'$, $x''y''z''$, $x'''y'''z'''$ be the given points: then by adopting the notation explained in the last number, the conditions of the question are

$$\begin{aligned} ax' + by' + cz' + m &= t', \\ ax'' + by'' + cz'' + m &= t'', \\ ax''' + by''' + cz''' + m &= t'''. \end{aligned}$$

Hence we obtain by elimination,

$$\begin{aligned} a(x't'' - x''t') + b(y't'' - y''t') + c(z't'' - z''t') + m(t'' - t') &= 0, \\ a(x''t''' - x'''t'') + b(y''t''' - y'''t'') + c(z''t''' - z'''t'') + m(t''' - t'') &= 0, \\ a(x'''t' - x't''') + b(y'''t' - y't''') + c(z'''t' - z't''') + m(t' - t''') &= 0, \end{aligned}$$

from which it appears (II) that the plane (2′) passes through three fixed points which lie on the three straight lines joining the given points. Now since these three points lie in the plane which passes through the three given points, and also in the plane (2′); they evidently lie on the straight line which is the common section of these two planes.

Hence, *if a system of surfaces of the second degree cut a given surface of the second degree along a given plane curve, and pass through three given points not on the given surface; the other planes in which they cut the given surface intersect each other in the same straight line.*

When the given points lie on the given surface, we have $t' = 0$, $t'' = 0$, $t''' = 0$; and the conditions become

$$\begin{aligned} ax' + by' + cz' + m &= 0, \\ ax'' + by'' + cz'' + m &= 0, \\ ax''' + by''' + cz''' + m &= 0; \end{aligned}$$

hence, in this case, the plane (2′) passes through the three given points, and is therefore absolutely fixed.

IV.

If a surface of the second degree pass through the intersection of a given surface of the second degree and a given plane, and through four given points, none of which lie on the given surface; the other plane in which it cuts the given surface passes through six fixed points, which lie in the straight lines joining the four given points.

If $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$, $x_4y_4z_4$ be the given points, we obtain, by adopting the notation of number II,

$$\begin{aligned} ax_1 + by_1 + cz_1 + m &= t_1, \\ ax_2 + by_2 + cz_2 + m &= t_2, \\ ax_3 + by_3 + cz_3 + m &= t_3, \\ ax_4 + by_4 + cz_4 + m &= t_4; \end{aligned}$$

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By taking these equations in combinations of twos, we should find that the plane (2′) passes through six points, which lie on the six straight lines joining the four given points (II).

When the given points lie in the same plane, the six fixed points evidently range upon the straight line in which that plane cuts the plane (2′).

The cases in which some of the given points are upon the given surface, and others not, are left as an exercise for the student.

V.

To find the equation of a surface of the second degree touching a given surface of the second degree along a given plane curve.

If $u=0$ be the equation of the given surface, $v=0$ the equation of the given plane, and $v_1=0$ that of any other plane section, the equation of a surface of the second degree passing through the intersections of these planes and the given surface will be $u=mvv_1$. Now when the two planes coincide, the two plane curves along which the surfaces intersect one another unite in a plane curve of contact. Hence the equation of the required surface is $u=mv^2$, or

$$u=(ax+by+cz+d)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

where $a=\alpha\sqrt{m}$, $b=\beta\sqrt{m}$, $c=\gamma\sqrt{m}$, $d=\sqrt{m}$. The plane $v=0$, or

$$ax+by+cz+d=0,$$

which passes through the curve of contact, may be called the *plane of contact* of the two surfaces; and the surfaces may be said to have a *plane contact* of the first order.

VI.

If each of two surfaces of the second degree touch a given surface of the second degree along a plane curve, the surfaces intersect each other upon two planes which pass through the intersection of the planes of contact; and any straight line which intersects these four planes is cut harmonically.

If $u=0$ denote the equation of the given surface of the second degree, and if

$$\left.\begin{aligned} ax+by+cz+d&=0\\ a'x+b'y+c'z+d'&=0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

be the equations of the planes of contact; the equations of the enveloping surfaces will be (v),

$$u=(ax+by+cz+d)^2,$$
$$u=(a'x+b'y+c'z+d')^2.$$

But, by taking the difference of these equations, we get

$$(a'x+b'y+c'z+d')^2-(ax+by+cz+d)^2=0,$$

which is equivalent to the simultaneous equations

$$\left.\begin{aligned} (a'+a)x+(b'+b)y+(c'+c)z+d'+d&=0\\ (a'-a)x+(b'-b)y+(c'-c)z+d'-d&=0\end{aligned}\right\} \quad \ldots\ldots\ldots (b)$$

hence the planes of contact (a) and the planes of intersection (b) form a system of harmonic radial planes, which intersect in the straight line represented by any two of these equations taken simultaneously.

This remarkable theorem is due to *Monge*.

VII.

If a surface of the second degree pass through two given points, and have a plane contact with a given surface of the second degree, the plane of contact will pass through a fixed point.

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Let $u = 0$ be the equation of the given surface;

$$ax + by + cz + d = 0$$

the equation of the plane of contact; then the equation of the enveloping surface will be (v),

$$u = (ax + by + cz + d)^2;$$

hence, if the given points be denoted by $x'y'z'$, $x''y''z''$, $x'''y'''z'''$, we shall have

$$ax' + by' + cz' + d = u'^{\frac{1}{2}},$$

$$ax'' + by'' + cz'' + d = u''^{\frac{1}{2}}.$$

From these equations we easily deduce

$$a(x'u''^{\frac{1}{2}} - x''u'^{\frac{1}{2}}) + b(y'u''^{\frac{1}{2}} - u''y'^{\frac{1}{2}}) + c(z'u''^{\frac{1}{2}} - z''u'^{\frac{1}{2}}) + d(u''^{\frac{1}{2}} - u'^{\frac{1}{2}}) = 0,$$

and therefore the plane of contact passes through the point whose co-ordinates are

$$\frac{x'u''^{\frac{1}{2}} - x''u'^{\frac{1}{2}}}{u''^{\frac{1}{2}} - u'^{\frac{1}{2}}}, \quad \frac{y'u''^{\frac{1}{2}} - y''u'^{\frac{1}{2}}}{u''^{\frac{1}{2}} - u'^{\frac{1}{2}}}, \quad \frac{z'u''^{\frac{1}{2}} - z''u'^{\frac{1}{2}}}{u''^{\frac{1}{2}} - u'^{\frac{1}{2}}};$$

which evidently lies on the straight line joining the two given points.

Hence, *in a system of surfaces of the second degree, if each of the surfaces pass through two given points and touch a given surface of the second degree along a plane curve (real or imaginary), the planes of contact (real or ideal) pass through the same point.*

VIII.

If a surface of the second degree pass through three given points, and have a plane contact with a given surface of the second degree; the plane of contact will pass through three fixed points which lie in the same straight line.

Let the equation of the indeterminate surface be (VII.)

$$ax + by + cz + d = u^{\frac{1}{2}};$$

then if $x'y'z'$, $x''y''z''$, $x'''y'''z'''$, be the given points, we obtain by the question

$$ax' + by' + cz' + d = u'^{\frac{1}{2}},$$

$$ax'' + by'' + cz'' + d = u''^{\frac{1}{2}},$$

$$ax''' + by''' + cz''' + d = u'''^{\frac{1}{2}};$$

hence, by following the process of number III, we find that the plane of contact,

$$ax + by + cz + d = 0,$$

passes through three fixed points which range in the same straight line.

Hence, *in a system of surfaces of the second degree, if each of the surfaces pass through three given points, and have a plane contact with a given surface of the second degree, the planes of contact intersect each other in the same straight line.*

IX.

To find the equation of a surface of the second degree, cutting a given surface of the second degree in a single plane curve (real or imaginary).

Let $u = 0$ denote the equation of the given surface of the second degree, and let

$$ax + by + cz + d = 0 \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

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be the equation of its plane section: then the equation of the required surface will be

$$u = ax + by + cz + d \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

For, it is evident that every point common to these two surfaces lies upon the plane (a).*

x.

If each of two surfaces of the second degree intersect a given surface of the second degree in a single plane curve, they intersect each other in a single plane curve, and the three planes of intersection pass through the same straight line.

If $u = 0$ denote the equation of the given surface; the equation of the other two surfaces, will be

$$u = ax + by + cz + d,$$
$$u = a'x + b'y + c'z + d';$$

and the equations of the planes of intersection, will be

$$ax + by + cz + d = 0,$$
$$a'x + b'y + c'z + d' = 0,$$
$$(a'-a)x + (b'-b)y + (c'-c)z + d'-d = 0:$$

consequently these planes intersect in the straight line represented by any two of the last three equations, considered as simultaneous.

xi.

The following theorems require only to be stated; the demonstrations being exactly similar to those given in Nos. VII. and VIII.

1°. *In a system of surfaces of the second degree, if each of the surfaces pass through two given points, and intersect a given surface of the second degree in a single plane curve; the planes of section (real or ideal) pass through the same point.*

2°. *In a system of surfaces of the second degree, if each of the surfaces pass through three given points, and intersect a given surface of the second degree in a single plane curve; the planes of section (real or ideal) pass through the same straight line.*

* This proposition might also be expressed as follows. To find the equation of a surface of the second degree, similar to a given surface of the second degree. The enunciation given in the text appears, however, to connect the proposition better with the other matters contained in this paper.

ON THE SUMMATION OF INFINITE SERIES.

[*Mr. Robert Rawson, Manchester.*]

(Continued from vol. i. p. 271.)

4. Let us now take $n = p = q = \ldots = 2$; then by theorem (2) we shall have

$$\frac{1}{(x-a)^2(x-b)^2(x-c)^2\ldots} = \frac{f(a)}{(x-a)^2} + \frac{f'(a)}{x-a} + \frac{f_1(b)}{(x-b)^2} + \frac{f_1'(b)}{x-b} + \frac{f_2(c)}{(x-c)^2} + \frac{f_2'(c)}{x-c} + \ldots\ldots\ldots\ldots\ldots (5)$$

The functions $f(a), f'(a), f_1(b)$, *etc.*, are readily obtained from the table in Art. 2.

Hence $$\int \frac{dx}{(x-a)^2(x-b)^2(x-c)^2\ldots} = f(a)\log(x-a) + f_1(b)\log(x-b) + f_2(c)\log(x-c) + \ldots - \frac{f'(a)}{x-a} - \frac{f_1'(b)}{x-b} - \frac{f_2'(c)}{x-c} - \ldots + C.$$

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If we again take a, b, c, etc, in arithmetical progression, that is, put $-a=a, -b=2a, -c=3a, etc.$, and observing that the relations in Art. 3 will obtain for the derived functions, we shall have

$$\frac{1}{(x+a)^2(x+2a)^2(x+3a)^2..}$$
$$=f(-a)\left\{\frac{1}{(x+a)^2}-\frac{1}{(x+ma)^2}\right\}+f'(-a)\left\{\frac{1}{x+a}-\frac{1}{x+ma}\right\}$$
$$+f_1(-2a)\left\{\frac{1}{(x+2a)^2}-\frac{1}{\{x+(m-1)a\}^2}\right\}+f_1'(-2a)\left\{\frac{1}{x+2a}-\frac{1}{x+(m-1)a}\right\}$$
$$+f_2(-3a)\left\{\frac{1}{(x+3a)^2}-\frac{1}{\{x+(m-2)a\}^2}\right\}+f_2'(-3a)\left\{\frac{1}{x+3a}-\frac{1}{x+(m-2)a}\right\}$$
$+ etc.$, to $\frac{m}{2}$ terms vertically ....(6)

This form was given by Mr. Woolhouse, (see *Ladies' Diary*, 1836), but in a somewhat less general manner, and therefore it is unnecessary to add examples in the summation of this class of series.

5. Let $n=p=q=..=3$; then by theorem (2) we shall have

$$\frac{1}{(x-a)^3(x-b)^3(x-c)^3..}=\frac{f(a)}{(x-a)^3}+\frac{f'(a)}{(x-a)^2}+\frac{f''(a)}{x-a}+\frac{f_1(b)}{(x-b)^3}+\frac{f_1'(b)}{(x-b)^2}$$
$$+\frac{f_1''(b)}{x-b}+\frac{f_2(c)}{(x-c)^3}+\frac{f_2'(c)}{(x-c)^2}+\frac{f_2''(c)}{x-c}, etc.$$

If, therefore, we put $a=0$; $-b=1$; $-c=2$; $-d=3$; $-e=4$, we get

$$\frac{1}{x^3(x+1)^3..(x+4)^3}=\frac{f(0)}{x^3}+\frac{f'(0)}{x^2}+\frac{f''(0)}{x}+\frac{f_1(-1)}{(x+1)^3}+\frac{f_1'(-1)}{(x+1)^2}$$
$$+\frac{f_1''(-1)}{x+1}+\frac{f_2(-2)}{(x+2)^3}+.....+\frac{f_4''(-4)}{x+4}$$
$$=\frac{1}{1^3.2^3.3^3.4^3}\left(\frac{1}{x^3}-\frac{25}{4x^2}+\frac{65}{3x}\right)+\frac{1}{1^3.2^3.3^3}\left\{\frac{1}{(x+1)^3}-\frac{5}{2(x+1)^2}+\frac{20}{3(x+1)}\right\}$$
$$+\frac{1}{1^3.2^3.2^3}\left\{\frac{1}{(x+2)^3}+\frac{15}{4(x+2)}\right\}-\frac{1}{1^3.2^3.3^3}\left\{\frac{1}{(x+3)^3}+\frac{5}{2(x+3)^2}+\frac{20}{3(x+3)}\right\}$$
$$+\frac{1}{1^3.2^3.3^3.4^3}\left\{\frac{1}{(x+4)^3}+\frac{25}{4(x+4)^2}+\frac{65}{3(x+4)}\right\};$$

which expression has been partly decomposed by Mr. Woolhouse, and completely so by Mr. Beecroft, but in a manner entirely different from the above.

When the roots are in arithmetical progression, this expression admits of being put in the same form as the one in the last article.

6. Now, if we take the particular case, $n=p=q=..$ Also $-a=a$; $-b=2a$; $-c=3a$, *etc.*, we have from (2)

$$\frac{1}{(x+a)^n(x+2a)^n(x+3a)^n...}$$
$$=\frac{f(-a)}{(x+a)^n}+\frac{f'(-a)}{(x+a)^{n-1}}+\frac{f''(-a)}{2(x+a)^{n-2}}+\frac{f'''(-a)}{2\cdot3(x+a)^{n-3}}\text{ to } n \text{ terms.}$$
$$+\frac{f_1(-2a)}{(x+2a)^n}+\frac{f_1'(-2a)}{(x+2a)^{n-1}}+\frac{f_1''(-2a)}{2(x+2a)^{n-2}}+\frac{f_1'''(-2a)}{2\cdot3(x+2a)^{n-3}}\text{ to } n \text{ terms.}$$

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$$+\frac{f_2(-3a)}{(x+3a)^n}+\frac{f_2'(-3a)}{(x+3a)^{n-1}}+\frac{f_2''(-3a)}{2(x+3a)^{n-2}}+\frac{f_2'''(-3a)}{2\cdot3(x+3a)^{n-3}}$$ to n terms.

$+$ *etc.*, to as many terms vertically as there are factors. .....(7)

The above equation, which is an easy deduction from equation (2), was, I believe, first given implicitly by Mr. Beecroft (see Ladies' Diary, 1843, page 94), who, it appears obtained it by induction.

Since
$$\frac{1}{a^n.(2a)^n.(3a)^n.....(ma)^n}=f(-a);$$
$$\frac{1}{(-a)^n.a^n.(2a)^n..\{(m-1)a\}^n}=f_1(-2a);$$
$$\frac{1}{(-2a)^n.(-a)^n.a^n..\{(m-2)a\}^n}=f_2(-3a);$$
$$\vdots \qquad\qquad \vdots$$
$$\frac{1}{(-ma)^n-..(-3a)^n.(-2a)^n.(-a)^n}=f_{m-1}(-ma),\ etc.$$

The cycle it appears is complete after m substitutions; consequently we have

$f(-a)=f_{m-1}(-ma)$; $f_1(-2a)=f_{m-2}\{-(m-1)a\}$; *etc.*
$f'(-a)=f'_{m-1}(-ma)$; $f_1'(-2a)=f'_{m-2}\{-(m-1)a\}$; *etc.*
$f_1''(-a)=f''_{m-1}(-ma)$; $f_1''(-2a)=f''_{m-2}\{-(m-1)a\}$; *etc.*
etc. *etc.*

Therefore equation (7) will, when n is even, take the following form

$$\frac{1}{(x+a)^n(x+2a)^n(x+3a)^n....(x+ma)^n}=$$
$$f(-a)\left\{\frac{1}{(x+a)^n}+\frac{1}{(x+ma)^n}\right\}$$
$$+f'(-a)\left\{\frac{1}{(x+a)^{n-1}}-\frac{1}{(x+ma)^{n-1}}\right\}$$
$$+f''(-a)\left\{\frac{1}{2(x+a)^{n-2}}+\frac{1}{2(x+ma)^{n-2}}\right\}$$ to n terms.
$$+f_1(-2a)\left\{\frac{1}{(x+2a)^n}+\frac{1}{\{x+(m-1)a\}^n}\right\}$$
$$+f_1'(-2a)\left\{\frac{1}{(x+2a)^{n-1}}-\frac{1}{\{x+(m-1)a\}^{n-1}}\right\}$$
$$+f_1''(-2a)\left\{\frac{1}{2(x+2a)^{n-2}}+\frac{1}{2\{x+(m-1)a\}^{n-2}}\right\}$$ to n terms.
$$+f_2(-3a)\left\{\frac{1}{(x+3a)^n}+\frac{1}{\{x+(m-2)a\}^n}\right\}$$
$$+f_2'(-3a)\left\{\frac{1}{(x+3a)^{n-1}}-\frac{1}{\{x+(m-2)a\}^{n-1}}\right\}$$
$$+f''_2(-3a)\left\{\frac{1}{2(x+3a)^{n-2}}+\frac{1}{2\{x+(m-2)a\}^{n-2}}\right\}$$ to n terms.

where $f(-a)$, $f_1(-2a)$, $f_2(-3a)$, *etc.*, must be continued to $\frac{m}{2}$ terms.

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A similar form may be readily obtained when n is an odd number.

This theorem, which will enable us to sum to infinity all series whose general term is $\frac{1}{(x+a)^n(x+2a)^n(x+3a)^n..(x+ma)^n}$, when n and m are any even numbers, is given implicitly by Mr. Beecroft, in the paper above referred to; who in order to use it was compelled to have recourse to a subsequent theorem which he gave without demonstration.

It will be seen however, that the above theorems are entirely free from this objection; the different functions being readily calculated from the table art 2.

For the summation of this class of series, see Mr. Beecroft's paper above-mentioned.

7. The following form of Mr. Woolhouse's theorem, will be found useful in the summation of series of the above nature. Let

$$A\Sigma_1^{\infty} f(x+a) = Af(1+a) + Af(2+a) + Af(3+a) \text{ etc. ad infinitum}$$

$$B\Sigma_1^{\infty} f(x+b) = Bf(1+b) + Bf(2+b) + Bf(3+b) \text{ etc. ad infinitum}$$

$$C\Sigma_1^{\infty} f(x+c) = Cf(1+c) + Cf(2+c) + Cf(3+c) \text{ etc. ad infinitum}$$

etc. etc.

Then we shall have

$$A\Sigma_1^{\infty} f(x+a) = A\Sigma_1^{\infty} f(x) - A\Sigma_1^{a} f(x); \quad B\Sigma_1^{\infty} f(x+b) = B\Sigma_1^{\infty} f(x) - B\Sigma_1^{b} f(x);$$

$$C\Sigma_1^{\infty} f(x+c) = C\Sigma_1^{\infty} f(x) - C\Sigma_1^{c} f(x), \text{ etc.};$$

therefore by addition, $A\Sigma_1^{\infty} f(x+a) + B\Sigma_1^{\infty} f(x+b) + C\Sigma_1^{\infty} f(x+c) ..$

$$= (A+B+C..)\Sigma_1^{\infty} f(x) - A\Sigma_1^{a} f(x) - B\Sigma_1^{b} f(x) - C\Sigma_1^{c} f(x) .. (9)$$

Therefore the summation, of the sums of the series whose general term is $f(x+a)$; $f(x+b)$; $f(x+c)$ *etc.* respectively, is made to depend upon the summation of the series whose general term is $f(x)$.

Hence we see, that there are two conditions which will enable us to sum to infinity series of this nature. 1st, When $A+B+C..=0$; 2nd, when $\Sigma_1^{\infty} f(x)$ can be summed. Now if $-A=B$ and $C=0$, *etc.* the above equation becomes

$$\Sigma_1^{\infty} f(x+a) - \Sigma_1^{\infty} f(x+b) = \Sigma_1^{b} f(x) - \Sigma_1^{a} f(x),$$

which when $b=0$ becomes the same expression as that given by Mr. Woolhouse.

8. As the decomposition of fractions has not been applied, as far as I know, to the summation of series, whose general term contains a given function of x in the numerator; the following summation, easily derived from the above theorems, may not be uninteresting.

$$\Sigma_1^{\infty}\left(\frac{(x+a')^2}{(x+a)^2(x+2a)^2}\right) = (A+A')\cdot\frac{\pi^2}{6} - A\Sigma_1^{a}\left(\frac{1}{x^2}\right) - A'\Sigma_1^{2a}\left(\frac{1}{x^2}\right) + B\Sigma_1^{a}\left(\frac{1}{x+a}\right)$$

.....(10)

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where $A=\frac{(a'-a)^2}{a^2}$; $A'=\frac{(a'-2a)^2}{a^2}$; $B=\frac{2}{a^3}\left\{3a'a-2a^2-a'^2\right\}$, a and a' being any whole numbers.

Take for example $a=1$, $a'=3$.

$$\therefore \frac{3^2}{1^2.2^2}+\frac{4^2}{2^2.3^2}+\frac{5^2}{3^2.4^2}+\frac{6^2}{4^2.5^2} \text{ etc., ad infinitum} = \frac{5}{6}\pi^2-4.$$

9. Required the decomposition of

$$\frac{\{x+(m+1)a\}^2\{x+(m+2)a\}^2..\{x+(2m-1)a\}^2}{(x+a)^2(x+2a)^2....(x+ma)^2}$$

Here, $-a=a$; $-b=2a$; $-c=3a$, *etc.*; consequently, equation (2) is

$$\frac{\{x+(m+1)a\}^2.\{x+(m+2)a\}^2....\{x+(2m-1)a\}^2}{(x+a)^2(x+2a)^2....(x+ma)^2}$$
$$=-\frac{f(-a)}{(x+a)^2}+\frac{f'(-a)}{x+a}$$
$$+\frac{f_1(-2a)}{(x+2a)^2}+\frac{f_1'(-2a)}{x+2a}$$
$$+\frac{f_2(-3a)}{(x+3a)^2}+\frac{f_2'(-3a)}{x+3a} \quad \ldots\ldots\ldots\ldots (11)$$

etc. *etc.* to m terms vertically.

where $f(x)=\frac{\{x+(m+1)a\}^2..\{x+(2m-1)a\}^2}{(x+2a)^2....(x+ma)^2}$;

$f_1(x)=\frac{\{x+(m+1)a\}^2..\{x+(2m-1)a\}^2}{(x+a)^2....(x+ma)^2}$; *etc.*

Now, if we find the values of $f(-a)$, $f'(-a)$; $f_1(-2a)$, $f_1'(-2a)$, *etc.*, we readily see that $f'(-a)+f_1'(-2a)+f_2'(-3a)..=0$; and therefore from equation (9) we shall have

$$\Sigma_1^{\infty}\left(\frac{\{x+(m+1)a\}^2..\{x+(2m-1)a\}^2}{(x+a)^2(x+2a)^2..(x+ma)^2}\right)$$
$$=\left\{f(-a)+f_1(-2a)+f_2(-3a)..f_{m-1}(-ma)\right\}.\Sigma_1^{\infty}\left(\frac{1}{x^2}\right)-f(-a)\Sigma_1^{a}\left(\frac{1}{x^2}\right)$$
$$-f_1(-2a)\,\Sigma_1^{2a}\left(\frac{1}{x^2}\right)-f_2(-3a)\,\Sigma_1^{3a}\left(\frac{1}{x^2}\right)-..f_{m-1}(-ma)\,\Sigma_1^{ma}\left(\frac{1}{x^2}\right)$$
$$-f'(-a)\,\Sigma_1^{a}\left(\frac{1}{x}\right)-f_1'(-2a)\,\Sigma_1^{2a}\left(\frac{1}{x}\right)-f_2'(-3a)\,\Sigma_1^{3a}\left(\frac{1}{x}\right)$$
$$-..f'_{m-1}(-ma)\,\Sigma_1^{ma}\left(\frac{1}{x}\right), \text{ etc.} \quad \ldots\ldots\ldots\ldots (12)$$

where

$$f(-a)=\frac{m^2.(m+1)^2..(2m-2)^2}{1^2.2^2.3^2...(m-1)^2};\ f_1(-2a)=\frac{(m-1)^2\,m^2..(2m-3)^2}{1^2.1^2.2^2.3^2...(m-2)^2};\ \text{etc.}$$

and $f'(-a)=\frac{2}{a}f(-a)\left\{\Sigma_m^{2m-2}\left(\frac{1}{m}\right)-\Sigma_1^{m-1}\left(\frac{1}{m}\right)\right\}$;

$$f_1'(-2a)=\frac{2}{a}f_1(-2a)\left\{\Sigma_{m-1}^{2m-3}\left(\frac{1}{m}\right)+\Sigma_1^{1}\left(\frac{1}{m}\right)-\Sigma_1^{2m-3}\left(\frac{1}{m}\right)\right\};$$

$$f_2'(-3a)=\frac{2}{a}f_2(-3a)\left\{\Sigma_{m-2}^{2m-4}\left(\frac{1}{m}\right)+\Sigma_1^{2}\left(\frac{1}{m}\right)-\Sigma_1^{2m-4}\left(\frac{1}{m}\right)\right\} \text{ etc.}$$

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Example 1st.

Sum the series $\frac{7^2.8^2.9^2.10^2}{2^2.3^2.4^2.5^2.6^2}+\frac{8^2.9^2.10^2.11^2}{3^2.4^2.5^2.6^2.7^2}+\frac{9^2.10^2.11^2.12^2}{4^2.5^2.6^2.7^2.8^2}$ *etc.* ad infinitum.

Here $a=1$, and $m=5$; therefore we shall have, $f(-a)=4900$; $f_1(-2a)=19600$; $f_2(-3a)=8100$; $f_3(-4a)=400$; $f_4(-5a)=1$; and $f'(-a)=-\frac{85190}{6}$; $f_1'(-2a)=-\frac{17360}{6}$; $f_2'(-3a)=\frac{92340}{6}$; $f_3'(-4a)=\frac{10160}{6}$; $f_4'(-5a)=\frac{50}{6}$;

$\therefore f(-a)+f_1(-2a)+f_2(-3a)+f_3(-4a)+f_4(-5a)=33001$, consequently by theorem (12) we shall have,

$$\Sigma_1^{\infty}\left(\frac{(x+6)^2..(x+9)^2}{(x+1)^2(x+2)^2(x+3)^2..(x+5)^2}\right)$$
$$=33001\,\Sigma_1^{\infty}\left(\frac{1}{x^2}\right)-4900\,\Sigma_1^{1}\left(\frac{1}{x^2}\right)-19600\,\Sigma_1^{2}\left(\frac{1}{x^2}\right)$$
$$-8100\,\Sigma_1^{3}\left(\frac{1}{x^2}\right)-400\,\Sigma_1^{4}\left(\frac{1}{x^2}\right)-\Sigma_1^{5}\left(\frac{1}{x^2}\right)$$
$$+\frac{85190}{6}\Sigma_1^{1}\left(\frac{1}{x}\right)+\frac{17360}{6}\Sigma_1^{2}\left(\frac{1}{x}\right)-\frac{92340}{6}\Sigma_1^{3}\left(\frac{1}{x}\right)-\frac{10160}{6}\Sigma_1^{4}\left(\frac{1}{x}\right)$$
$$-\frac{50}{6}\Sigma_1^{5}\left(\frac{1}{x}\right)=33001.\frac{\pi^2}{6}-\frac{195189769}{3600}.$$

Example 2. Sum the series

$$\frac{8^2.9^2.10^2.11^2.12^2}{2^2.3^2.4^2.5^2.6^2.7^2}+\frac{9^2.10^2.11^2.12^2.13^2}{3^2.4^2.5^2.6^2.7^2.8^2}+\frac{10^2.11^2.12^2.13^2.14^2}{4^2.5^2.6^2.7^2.8^2.9^2}+\text{ad infinitum.}$$

Here $a=1$, and $m=6$; $f(-a)=63504$; $f_1(-2a)=396900$; $f_2(-3a)=313600$; $f_3(-4a)=44100$; $f_4(-5a)=900$, $f_5(-6a)=1$; $f'(-a)=\frac{-1040004}{5}$, $f_1'(-2a)=-268065$; $f_2'(-3a)=\frac{1037120}{3}$, $f_3'(-4a)=125790$, $f_4'(-5a)=2560$; $f_5'(-6a)=\frac{137}{15}$;

$\therefore f(-a)+f_1(-2a)+..f_5(-6a)=819005$; therefore by (12) we shall have

$$\Sigma_1^{\infty}\left\{\frac{(x+7)^2(x+8)^2..(x+11)^2}{(x+1)^2(x+2)^2..(x+6)^2}\right\}$$
$$=819005\,\Sigma_1^{\infty}\left(\frac{1}{x^2}\right)-63504\,\Sigma_1^{1}\left(\frac{1}{x^2}\right)-396900\,\Sigma_1^{2}\left(\frac{1}{x^2}\right)$$
$$-313600\,\Sigma_1^{3}\left(\frac{1}{x^2}\right)-44100\,\Sigma_1^{4}\left(\frac{1}{x^2}\right)-900\,\Sigma_1^{5}\left(\frac{1}{x^2}\right)-\Sigma_1^{6}\left(\frac{1}{x^2}\right)$$
$$+\frac{1040004}{5}\Sigma_1^{1}\left(\frac{1}{x}\right)+268065\,\Sigma_1^{2}\left(\frac{1}{x}\right)-\frac{1037120}{3}\Sigma_1^{3}\left(\frac{1}{x}\right)-125790\,\Sigma_1^{4}\left(\frac{1}{x}\right)$$
$$-4560\,\Sigma_1^{5}\left(\frac{1}{x}\right)-\frac{137}{15}\Sigma_1^{6}\left(\frac{1}{x}\right)=819005.\frac{\pi^2}{6}-\frac{3782064369}{3600}-\frac{1066227140}{360}$$
$$=819005.\frac{\pi^2}{6}-\frac{4848291509}{3600}.$$

Scho. The application of the theorem art. 7, to the theorem (2) art. 1, would enable us to give the complete theory of the summation of infinite series by means of the decomposition of rational fractions.

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where $A=\frac{(a'-a)^2}{a^2}$; $A'=\frac{(a'-2a)^2}{a^2}$; $B=\frac{2}{a^3}\left\{3a'a-2a^3-a'^3\right\}$, a and a' being any whole numbers.

Take for example $a=1$, $a'=3$.

$$\therefore \frac{3^2}{1^2.2^2}+\frac{4^2}{2^2.3^2}+\frac{5^2}{3^2.4^2}+\frac{6^2}{4^2.5^2} \text{ etc., ad infinitum} = \frac{5\pi^2}{6}-4.$$

9. Required the decomposition of

$$\frac{\{x+(m+1)a\}^2\{x+(m+2)a\}^2..\{x+(2m-1)a\}^2}{(x+a)^2(x+2a)^2....(x+ma)^2}$$

Here, $-a=a$; $-b=2a$; $-c=3a$, *etc.*; consequently, equation (2) is

$$\frac{\{x+(m+1)a\}^2.\{x+(m+2)a\}^2....\{x+(2m-1)a\}^2}{(x+a)^2(x+2a)^2....(x+ma)^2}$$
$$=-\frac{f(-a)}{(x+a)^2}+\frac{f'(-a)}{x+a}$$
$$+\frac{f_1(-2a)}{(x+2a)^2}+\frac{f_1'(-2a)}{x+2a}$$
$$+\frac{f_2(-3a)}{(x+3a)^2}+\frac{f_2'(-3a)}{x+3a} \quad \ldots\ldots\ldots\ldots (11)$$

etc. *etc.* to m terms vertically.

where $f(x)=\frac{\{x+(m+1)a\}^2..\{x+(2m-1)a\}^2}{(x+2a)^2....(x+ma)^2}$;

$$f_1(x)=\frac{\{x+(m+1)a\}^2..\{x+(2m-1)a\}^2}{(x+a)^2....(x+ma)^2}; \text{ etc.}$$

Now, if we find the values of $f(-a)$, $f'(-a)$; $f_1(-2a)$, $f_1'(-2a)$, *etc.*, we readily see that $f'(-a)+f_1'(-2a)+f_2'(-3a)..=0$; and therefore from equation (9) we shall have

$$\Sigma_1^{\infty}\left(\frac{\{x+(m+1)a\}^2..\{x+(2m-1)a\}^2}{(x+a)^2(x+2a)^2..(x+ma)^2}\right)$$
$$=\left\{f(-a)+f_1(-2a)+f_2(-3a)..f_{m-1}(-ma)\right\}.\Sigma_1^{\infty}\left(\frac{1}{x^2}\right)-f(-a)\Sigma_1^{a}\left(\frac{1}{x^2}\right)$$
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$$-..f'_{m-1}(-ma)\,\Sigma_1^{ma}\left(\frac{1}{x}\right), \text{ etc.} \quad \ldots\ldots\ldots\ldots (12)$$

where

$$f(-a)=\frac{m^2.(m+1)^2..(2m-2)^2}{1^2.2^2.3^2...(m-1)^2}; f_1(-2a)=\frac{(m-1)^2\,m^2..(2m-3)^2}{1^2.1^2.2^2.3^2...(m-2)^2}; \text{ etc.}$$

and $f'(-a)=\frac{2}{a}f(-a)\left\{\Sigma_m^{2m-2}\left(\frac{1}{m}\right)-\Sigma_1^{m-1}\left(\frac{1}{m}\right)\right\}$;

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Example 1st.

Sum the series $\frac{7^2.8^2.9^2.10^2}{2^2.3^2.4^2.5^2.6^2}+\frac{8^2.9^2.10^2.11^2}{3^2.4^2.5^2.6^2.7^2}+\frac{9^2.10^2.11^2.12^2}{4^2.5^2.6^2.7^2.8^2}$ *etc.* ad infinitum.

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$$-4560\,\Sigma_1^{5}\left(\frac{1}{x}\right)-\frac{137}{15}\Sigma_1^{6}\left(\frac{1}{x}\right)=819005.\frac{\pi^2}{6}-\frac{3782064369}{3600}-\frac{1066227140}{360}$$
$$=819005.\frac{\pi^2}{6}-\frac{4848291509}{3600}.$$

Scho. The application of the theorem art. 7, to the theorem (2) art. 1, would enable us to give the complete theory of the summation of infinite series by means of the decomposition of rational fractions.

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The discussion of this theory, however, I shall reserve till some future opportunity. In conclusion I shall notice, briefly, the application of this theorem to the summation of those fractions whose factors have the same exponent and whose roots are in arithmetical progression.

10. Applying (9) to the summation of (7) we shall have

$$\Sigma_i^{\infty} \frac{1}{(x+a)^n(x+2a)^n..(x+ma)^n} = \{f(-a)+..f_{m-1}(-ma)\}\ \Sigma_1^{\infty}\left(\frac{1}{x^n}\right)$$

$$-f(-a)\,\Sigma_1^{a}\left(\frac{1}{x^n}\right)-f_1(-2a)\,\Sigma_1^{2a}\left(\frac{1}{x^n}\right)-..f_{m-1}(-ma)\Sigma_1^{ma}\left(\frac{1}{x^n}\right)$$

$$+\{f'(-a)+f_1'(-2a)+..f'_{m-1}(-ma)\}\Sigma_1^{\infty}\left(\frac{1}{x^{n-1}}\right)$$

$$-f'(-a)\ \Sigma_1^{a}\left(\frac{1}{x^{n-1}}\right)-f_1'(-2a)\,\Sigma_1^{2a}\left(\frac{1}{x^{n-1}}\right)-..f'_{m-1}(-ma)\Sigma_1^{ma}\left(\frac{1}{x^{n-1}}\right)$$

$$+\left\{\frac{f''(-a)}{2}+..\frac{f''_{m-1}(-ma)}{2}\right\}\Sigma_1^{\infty}\left(\frac{1}{x^{n-2}}\right)-\frac{f''(-a)}{2}\ \Sigma_1^{a}\left(\frac{1}{x^{n-2}}\right)$$

$$-\frac{f_1''(-2a)}{2}\Sigma_1^{2a}\left(\frac{1}{x^{n-2}}\right)-..\frac{f''_{m-1}(-ma)}{2}\Sigma_1^{ma}\left(x^{n-2}\right)\text{to } n \text{ terms}....(13)$$

Now, it appears that $f^{n-1}(-a)+f_1^{n-1}(-2a)+..f_{m-1}^{n-1}(-ma)=0$, must be satisfied, for if not, the series whose general term is $\frac{1}{(x+a)^n..(x+ma)^n}$ will be divergent, and therefore its sum will be ∞.

The three theorems, given by Mr. Beecroft, (see Ladies' Diary, 1844, page 87, 88) are therefore contained in the above equation.

I may, however, add that neat results may be obtained by taking particular values of n in the above equation.

A PROPERTY RELATIVE TO SURFACES WHICH PASS THROUGH THE SAME EIGHT POINTS IN SPACE.

[*Mr. Fenwick.*]

It is well known both to English and continental Mathematicians, that an infinite number of curves of the second degree can be made to pass through four points in a plane, and that if in each of these there be drawn the diameter of the chords that are parallel to a given line, all the diameters intersect in the same point.

The existence in space of a theorem analogous to this, has been both affirmed and denied by mathematicians of great eminence. My object, therefore, is to inquire whether there be in space a property exactly analogous to this in plano, or whether it exist only for certain positions of the points, through which all the surfaces pass, or for certain positions of the plane, to which all the diameters are conjugate.

The four points in a plane evidently correspond to eight points in space. Let the following then be two of the infinite number of surfaces of the second degree that can be made to pass through the same eight points:—

$$u=ax^2+by^2+cx^2+2dxy+2exz+2fyz+2gz+2hy+2kx+1=0..(1)$$
$$u'=a'x^2+b'y^2+c'x^2+2d'xy+2e'xz+2f'yz+2g'z+2h'y+2k'x+1=0..(2)$$

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Then λ being an arbitrary multiplier, *any third surface* passing through the same eight points is

$$u + \lambda u' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Hence the equations of a diameter in reference to the surface (3), conjugate to a plane $z = my + nx + p$, are (*Mathematician*, vol. 1. p. 239.)

$$(f+\lambda f')z + (b+\lambda b')y + (d+\lambda d')x + (h+\lambda h') + m\{(a+\lambda a')z + (f+\lambda f')y + (e+\lambda e')x + (g+\lambda g')\} = 0,$$

$$(e+\lambda e')z + (d+\lambda d')y + (c+\lambda c')x + (k+\lambda k') + n\{(a+\lambda a')z + (f+\lambda f')y + (e+\lambda e)x + (g+\lambda g')\} = 0.$$

These, in consequence of the indeterminate quantity, λ, readily reduce to the following,

$$(f+ma)z + (b+mf)y + (d+me)x + (h+mg) = 0 \quad \ldots\ldots\ldots (4)$$
$$(f'+ma')z + (b'+mf')y + (d'+me')x + (h'+mg') = 0 \quad \ldots\ldots\ldots (5)$$
$$(e+na)z + (d+nf)y + (c+ne)x + (k+ng) = 0 \quad \ldots\ldots\ldots (6)$$
$$(e'+na')z + (d'+nf')y + (c'+ne')x + (k'+ng') = 0 \quad \ldots\ldots\ldots (7)$$

Now if the same values of x, y, z satisfy all these equations, the diameters will obviously pass through the same point.

Since the equations (4, 5, 6, 7) can be combined three and three in four different ways, we can deduce the values of x, y, z from them in four different ways, viz., from (4, 5, 6), (4, 5, 7), (5, 6, 7) and (4, 6, 7). If we use the first set of equations, c' and k' will be wanting in the resulting expressions for x, y, z; if the second, c and k; if the third, b and h; and if the fourth, b' and h'. But since the quantities a, a', b, b', *etc.*, are absolutely independent of each other, the values of x, y, z thus obtained, will be all different, and hence, *generally*, the diameters do not pass through the same point.

Let us next inquire whether the diameters pass through the same point for certain positions of the eight points.

If we eliminate x, y, z from the equations (4, 5, 6, 7), we shall find that the resulting *equation of condition* for which the diameters meet, is satisfied by either of the following relations (the equation of condition is omitted on account of its great length),

$$\left.\begin{array}{l} a = a',\ c = c',\ d = d',\ e = e',\ f = f',\ g = g',\ k = k' \\ a = a',\ b = b',\ d = d',\ e = e',\ f = f',\ g = g',\ h = h' \end{array}\right\} \ldots\ldots (8)$$

Taking the former of these, the equations (1) and (2) become

$$ax^2 + by^2 + cx^2 + 2dxy + 2exz + 2fyz + 2gz + 2hy + 2kx + 1 = 0,$$
$$ax^2 + b'y^2 + cx^2 + 2dxy + 2exz + 2fyz + 2gz + 2h'y + 2kx + 1 = 0.$$

The traces of these on the co-ordinate plane of xz are the same, viz.,

$$ax^2 + cx^2 + 2exz + 2gz + 2kx + 1 = 0.$$

One of the intersections of the surfaces is therefore a plane curve in the plane of xz.

Again, taking (2) from (1) and applying the former of the relations (8), we get

$$(b - b')y^2 + 2(h - h')y = 0 \quad \ldots\ldots\ldots\ldots\ldots (9)$$

This, which gives a relation between the co-ordinates of the lines of intersection, resolves into the equations

$$y = 0, \text{ and } (b - b')y + 2(h - h') = 0.$$

The former of these gives the plane of xz, or the plane of intersection which we have noticed; the latter, being the equation of a plane parallel to the plane of xz, shows that a second intersection of the surfaces (if there be one) is also a plane curve in a plane parallel to that of the other intersection.

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The equation of condition for which the diameters meet, is also satisfied by $h = h'$, in addition to the relations (8). This reduces (9) to $y^2 = 0$, which implies that the lines of intersection coincide, or that the surfaces touch one another in a plane curve.

Similar results are obtained when the second of the relations (8) is employed. Hence

(1) *If all surfaces of the second degree which have eight points common, be cut by planes parallel to a given plane, the diameters conjugate to these sections will all pass through the same point, when the eight points are either in two parallel planes, or in the same plane.*

Next, let the given plane be parallel to one of the co-ordinate planes, that of xy suppose, or $m = 0$, and $n = 0$. The equations therefore of the diameters reduce to

$$\begin{aligned} fz + by + dx + h &= 0, \\ f'z + b'y + d'x + h' &= 0, \\ ez + dy + cx + k &= 0, \\ e'z + d'y + c'x + k' &= 0. \end{aligned}$$

Eliminating x, y, z, from these, we have

$$\begin{aligned} &(df' - d'f)\{b(ke' - k'e) + h(ed' - e'd) + f(dk' - d'k)\} \\ &+ (hf' - h'f)\{d(de' - d'e) + b(ec' - e'c) + f(cd' - c'd)\} \\ &+ (bf' - b'f)\{d(ek' - e'k) + h(ce' - c'e) + f(kc' - k'c)\} \\ &= 0. \end{aligned}$$

This equation of condition for which the diameters meet, is satisfied by either of the following relations,

$$\left.\begin{aligned} b = b', d = d', f = f', h = h' \\ \text{or} \quad c = c', d = d', e = e', k = k' \end{aligned}\right\} \ldots\ldots\ldots\ldots (10)$$

The equations (1) and (2) by means of the former of these become

$$\begin{aligned} az^2 + by^2 + cx^2 + 2dxy + 2exz + 2fyz + 2gz + 2hy + 2kx + 1 &= 0, \\ a'z^2 + by^2 + c'x^2 + 2dxy + 2e'xz + 2fyz + 2g'z + 2hy + 2k'x + 1 &= 0. \end{aligned}$$

Taking the latter from the former we get

$$\begin{aligned} (a - a')z^2 + (c - c')x^2 + 2(e - e')xz + 2(g - g')z \\ + 2(k - k')x = 0. \end{aligned}$$

This being the equation of a cylindrical surface parallel to the given plane, shows that the eight points are on a cylindrical surface whose generator is parallel to the given plane, and whose director is a plane curve.

The second of the relations (10) gives a similar result.

Hence,

(2) *The diameters conjugate to sections made by planes parallel to a given plane will also pass through the same point, when the eight points that are common to all the surfaces lie on a cylindrical surface parallel to the given plane, whose director is a plane curve of the second degree.*

The theorem *in general* was proposed in the "Lady's and Gentleman's Diary" for 1843, but no demonstration of it has been given. It is very probable, that many of those theorems in plano that might be supposed to have corresponding properties in space, in respect of lines meeting in a point, fail in a similar way as to the general case. In a plane, two lines *necessarily* intersect one another, either at a finite or an infinite distance. This, however, is not the case in space, for the lines may be in different planes.

August, 7th, 1845.

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NEW DEMONSTRATIONS OF THE THEOREMS OF PASCAL AND BRIANCHON.

[*Mr. Thomas Weddle, Newcastle-on-Tyne.*]

If $X = 0..(1)$, $Y = 0..(2)$, $Z = 0..(3)$,
and $X = bY + cZ..(4)$, $X = b_1Y + c_1Z..(5)$, $X = b_2Y + c_2Z..(6)$,
be the equations to the sides of two triangles, and, if these triangles be either inscribed in or described about, the same conic section, there will be a relation among the constants b, c, b_1, *etc.*, which may be found (Mathematician, vol i. p. 327—8) by eliminating M and N from the equations

$$bc + Mc + Nb = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$
$$b_1c_1 + Mc_1 + Nb_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$
$$b_2c_2 + Mc_2 + Nb_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

Let ABCDEF be a hexagon inscribed in a conic section, whose sides AB, DE intersect in P; BC, EF in Q, and CD, FA in R. Let (1, 2, 3) be the equations to the sides CA, AB, BC of the triangle ABC, and (4, 5, 6), those to the sides FD, DE and EF of the triangle DEF. It follows from (1, 3) and (1, 2), that, if Y be eliminated from (4, 5), and Z from (4, 6), we shall have the equations to CD and FA, which are, therefore

$$(b_1 - b)X = (b_1c - bc_1)Z$$
$$(c_2 - c)X = (bc_2 - b_2c)Y$$

Now (7, 8) and (7, 9) give, $(b_1 - b)cc_1 + (b_1c - bc_1)N = 0$, and $(c_2 - c)bb_2 + (bc_2 - b_2c)M = 0$; hence the equations of CD and FA become,

$$NX + cc_1Z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$
and,
$$MX + bb_2Y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

The equations to the sides AB, BC, CD, DE, EF and FA of the hexagon are therefore (2, 3, 10, 5, 6, 11). The equation of any line through P may (2, 5) be denoted by,

$$mX + nY - mc_1Z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

and (10, 11) that of any line through R, by

$$(Mp + Nq)X + bb_2pY + cc_1qZ = 0 \quad \ldots\ldots\ldots\ldots (13)$$

If (12, 13) coincide, each of them will be the equation of PR; hence equating the coefficients, (which we are at liberty to do, since, to avoid fractions, we have introduced *four* multipliers m, n, p, q instead of *two*,) we get,

$$m = Mp + Nq, \quad n = bb_2p, \text{ and } -mc_1 = cc_1q.$$

Eliminating m we get, $Mp = -(N + c)q = (7)\ \frac{Mc}{b}q$ or $bp = cq$,

hence we may take $p = c$, $q = b$, $\therefore m = -bc$ and $n = bb_2c$, and (12) or (13) becomes,
$$X = b_2Y + c_1Z \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$
which is the equation of PR. By proceeding in a similar manner we shall find the very same equation for that of QR; hence PR, QR, coincide, and P, Q, R are in a straight line; or,

(A). *If the opposite sides of a hexagon inscribed in a conic section be produced to meet, the three points of intersection range in a straight line.*

Again, let ABCDEF be a hexagon circumscribing a conic section. If the alternate sides be produced, two triangles will be formed circumscribing the curve; let then (1, 2, 3) be the equations to AB, CD and EF, and (4, 5, 6) those to BC, DE and FA. By (1, 6) the equation of any line through A is, $mX = b_2Y + c_2Z$, and (2, 5) that of any line through D, is, $X = nY + c_1Z$.

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If these coincide, each of them will represent the diagonal AD, $\therefore m = \frac{b_2}{n} = \frac{c_2}{c_1}$. Hence the equation of AD, is,

$$X = \frac{c_1}{c_2}(b_2Y + c_2Z) \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

In the same manner we find the equations of BE and CF to be

$$X = \frac{b_1}{b}(bY + cZ) \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

$$X = b_2Y + cZ \quad \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

Multiply (15) by Nb_1, (16) by Mc_1, and (17) by b_1c_1, and add; the coefficients X, Y, Z, vanish by (7, 8, 9); consequently (15, 16, 17) can be satisfied by the same values of X, Y, Z; and, hence, the diagonals AD, BE, and CF intersect each other in the same point. This theorem may be thus enunciated;

(B). *The diagonals joining the opposite angular points of a hexagon circumscribing a conic section, pass through the same point.*

The property (A) is the well known theorem of Pascal, and (B) that of Brianchon. The preceding demonstrations of these elegant properties will, I trust, be found to possess originality, and perhaps some degree of elegance. They were suggested to me by my solution to exercise 37 of the present work (vol i. p. 327—8).

MATHEMATICAL NOTE.

The points of intersection of all the conjugate tangents of an ellipse lie in the circumference of a concentric ellipse, the axes of which are diagonals of the squares described on the axes of the original ellipse.

Definition. The touching lines which are drawn through the extremities of two conjugate diameters of the ellipse are named in this investigation *conjugate tangents*.

Let $y = px + q$ ..... (1), and $y = p_1x + q_1$ ..... (2), be two conjugate tangents of an ellipse referred to its principal axes, and denoted by the equation

$$a^2y^2 + b^2x^2 = a^2b^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Then the condition that (1) and (2) are conjugate is (*Mathematician*, vol. i., p. 130),

$$a^2pp_1 + b^2 = 0, \text{ or } p_1 = -\frac{b^2}{pa^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Eliminating y between (1) and (3), and finding the condition of equal roots, so that (1) may be tangential to (3), we have

$$q^2 = a^2p^2 + b^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Similarly, $$q_1^2 = a^2p_1^2 + b^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Hence, we readily get from (1, 5)

$$\{p(x^2 - a^2) - xy\}^2 = b^2(x^2 - a^2) + a^2y^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Also from (2, 4, 6)

$$\{pa^2(y^2 - b^2) + b^2xy\}^2 = b^4\{a^2(y^2 - b^2) + b^2x^2\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Now the co-ordinates of the intersections of the tangents must satisfy (7) and (8), p being supposed to have all values. Eliminating, therefore, p between these equations, we get, after slight reductions,

$$a^2(y^2 - b^2) + b^2(x^2 - a^2) = 0,$$

as the equation of the locus of the intersections of the tangents, which is evidently the equation of an ellipse concentric with (3). Its semi-axes are $a\sqrt{2}$ and $b\sqrt{2}$.

F.

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COMPUTATION OF LOGARITHMS AND ANTILOGARITHMS.

[*Mr. Weddle.*]

1. Before proceeding to the consideration of the processes about to be offered to the perusal of the readers of the "Mathematician," it will be convenient to introduce the tables which will be required, and to explain the manner in which these may be constructed.

TABLE I. M and multiples.

M =	·4342944819032518
Double =	·8685889638065037
Triple =	1·3028834457097555
4ple =	1·7371779276130073
5ple =	2·1714724095162591
6ple =	2·605766891419511 0
7ple =	3·0400613733227628
8ple =	3·4743558552260146
9ple =	3·9086503371292664

TABLE II. $\frac{1}{M} = \log_e 10$ and multiples.

$\frac{1}{M}$ =	2·3025850929940457
	4·6051701859880914
	6·9077552789821371
	9·2103403719761827
	11·5129254649702284
	13·8155105579642741
	16·1180956509583198
	18·4206807439523655
	20·7232658369464112

TABLE III.

NO.	NAPIERIAN LOG.	COMMON LOG.
2	·6931471805599453	·3010299956639812
3	1·0986122886681097	·4771212547196624
4	1·3862943611198906	·6020599913279624
5	1·6094379124341004	·6989700043360188
6	1·7917594692280550	·7781512503836436
7	1·9459101490553133	·8450980400142568
8	2·0794415416798359	·9030899869919436
9	2·1972245773362194	·9542425094393249

TABLE IV.

RECIP. OF NO.	NAPIERIAN LOG.	COMMON LOG.
1—·4	·5108256237659907	·2218487496163564
·3	·3566749439387324	·1549019599857432
·2	·2231435513142098	·0969100130080564
·1	·1053605156578263	457574905606751
.09	·0943106794712413	409586076789064
8	833816089390511	362121726544447
7	725706928348354	315170514460649
6	618754037180875	268721464003013
5	512932943875505	222763947111522
4	408219945202551	177287669604316
3	304592074847085	132282657337551
2	202027073175194	0087739243075051
1	100503358535014	43648054024501

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RECIP. OF NO.	NAPIERIAN LOG.	COMMON LOG.
1—·009	0090407446521491	39263455147248
8	80321716972643	3488327845813
7	70246149369645	30507515046188
6	60180723255630	26136156026867
5	50125418235443	21769192542745
4	40080213975388	17406615763013
3	30045090202987	13048416883443
2	20020026706731	0008694587126289
1	10005003335835	4345117740177
·0009	0009004052431641	3910410285829
8	8003201707691	3475746339209
7	7002451143934	3041125891608
6	6001800720324	2606548934320
5	5001250416823	2172015458643
4	4000800213397	1737525455876
3	3000450090020	1303078917322
2	2000200026671	0000868675834286
1	1000050003334	434316198075
·00009	0000900040502430	390882623695
8	800032001707	347449483687
7	700024501143	304016778044
6	600018000720	260584506755
5	500012500417	217152669814
4	400008000213	173721267210
3	300004500090	130290298935
2	200002000027	0000086859764981
1	100000500003	43429665339
·000009	0000090000405002	39086679262
8	80000320002	34743697527
7	70000245001	30400720136
6	60000180001	26057747088
5	50000125000	21714778382
4	40000080000	17371814020
3	30000045000	13028854000
2	20000020000	0000008685898324
1	10000005000	4342946991
0000009	0000009000004050	3908652096
8	8000003200	3474357245
7	7000002450	3040062437
6	6000001800	2605797673
5	5000001250	2171472952
4	4000000800	1737178275
3	3000000450	1302883641
2	2000000200	0000000868589051
1	1000000050	434294504
00000009	0000000900000041	390865051
8	800000032	347435599
7	700000025	304006148
6	600000018	260576697
5	500000013	217147246
4	400000008	173717796
3	300000005	130288347
2	200000002	0000000086858897
1	100000001	43429448

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Tables I. and II. consist of the modulus (M), and of the reciprocal of the modulus, together with their multiples as far as 9. The second and third columns of Table III. contain the Napierian and common logarithms of the numbers in the first column, while the logarithms in Table IV. are not those of the numbers in the first column, but of their reciprocals.

2. If the logarithms of 2, 3, 5, and 7 be found by some of the methods proposed in Professor Young's excellent little treatise on the Computation of Logarithms, we shall, by addition or subtraction merely, obtain the reciprocal of the Modulus, the logarithms of the numbers in Table III., and also the logarithms of $\frac{1}{1-\cdot4}$, $\frac{1}{1-\cdot3}$, $\frac{1}{1-\cdot2}$, and $\frac{1}{1-\cdot1}$ in Table IV.

The remainder of Table IV. is to be calculated by the formula (a) below. I shall, for the sake of easy reference, arrange the numbers in the first column in nine horizontal rows :—

1—·01	1—·001	1—·0001.....
1—·02	1—·002	1—·0002.....
1—·03	1—.003	1—·0003.....
1—·04	1—·004	1—·0004.....
1—·05	1—·005	1—·0005.....
1—·06	1—·006	1—·0006.....
1—·07	1—·007	1—·0007.....
1—·08	1—·008	1—·0008.....
1—·09	1—·009	1—·0009.....

3. $\text{Log}_e \frac{1}{1-x} = -\log_e (1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + etc.$, let $x = \frac{a}{10^m}$

$$\therefore \text{Log}_e \frac{1}{1-\frac{a}{10^m}} = \frac{a}{10^m} + \frac{1}{2}\frac{a^2}{10^{2m}} + \frac{1}{3}\frac{a^3}{10^{3m}} + etc.$$

$$= \text{A}.\frac{1}{10^m} + \text{B}.\frac{1}{10^{2m}} + \text{C}.\frac{1}{10^{3m}} + etc \ \ldots\ldots\ (a)$$

where $\text{A} = a$, $\text{B} = \frac{1}{2}a^2$, $\text{C} = \frac{1}{3}a^3$, *etc.*

If in (a) we make $a = 1$, and m successively $= 2, 3, 4, \ldots$ we shall get the logarithms of the reciprocals of the numbers in the first of the above rows.

If $a = 2$, and m successively $= 2, 3, 4, \ldots$, we shall obtain the logarithms of the reciprocals of the numbers in the second row.

⋮

If $a = 9$, and m successively $= 2, 3, 4, \ldots$ we shall get the logarithms of the reciprocals of the numbers in the ninth or last row.

The Napierian logarithms having been thus found, the common logarithms may be derived by multiplying by the modulus; they were however calculated independently by taking in (a), $\text{A} = a\text{M}$, $\text{B} = \frac{1}{2}a^2\text{M}$, $\text{C} = \frac{1}{3}a^3\text{M}$, *etc.*, and proceeding as directed for Napierian logarithms. The latter seems the preferable plan.

Since A, B, C, *etc.* are independent of m, it is obvious that if these quantities be calculated for the purpose of computing the logarithm of the reciprocal of the first number in any of the preceding rows, the logarithms of the reciprocals of the remaining numbers in that row will be found by

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addition merely, after removing the initial figures of A, B, C, *etc.* a certain number of places to the right. It hence follows that the construction of Table IV. may be effected with great ease.

Having now explained the construction of the tables, we are prepared to proceed to the immediate object of this paper.

4. Let it be required to find the logarithm of any number N to p places of decimals.

Reduce N to a number between 1 and 2 by placing the decimal point after the first significant figure and then dividing by the initial or any convenient digit (b) that will cause the integral part to become unity. Denote the quotient by $1+n$. Divide n the decimal part of $1+n$, by $1+n$, and let the quotient be r_1, and remainder n_1; divide n_1 by $1+n_1$, and let the quotient be r_2 and remainder n_2; repeat this process until the remainder (n_m) begins with at least $\frac{1}{3}p$ ciphers. (Each of the quotients $r_1, r_2 \ldots r_m$ must be restricted to one significant figure).

It is evident, that,

$$n_1 = n - r_1(1+n) \quad \text{or} \quad 1+n = \frac{1}{1-r_1}(1+n_1)$$

$$n_2 = n_1 - r_2(1+n_1) \quad \text{or} \quad 1+n_1 = \frac{1}{1-r_2}(1+n_2)$$

$$\vdots$$

$$n_m = n_{m-1} - r_m(1+n_{m-1}) \text{ or } 1+n_{m-1} = \frac{1}{1-r_m}(1+n_m).$$

Multiply the right hand equations, and cancel the common factors,

$$\therefore\ 1+n = \frac{1}{1-r_1}\cdot\frac{1}{1-r_2}\cdot\frac{1}{1-r_3}\ldots\ldots\frac{1}{1-r_m}\cdot(1+n_m)$$

Take the logarithm of each side,

$$\therefore \log(1+n) = \log\frac{1}{1-r_1} + \log\frac{1}{1-r_2} + \log\frac{1}{1-r_3}\ldots\ldots$$

$$+\log.\frac{1}{1-r_m} + \log(1+n_m) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If μ be the modulus, we have $\log(1+n_m) = \mu\{n_m - \frac{1}{2}n_m^2 + \frac{1}{3}n_m^3\ etc.\}$; now, n^m is a decimal beginning with not less than $\frac{1}{2}p$ ciphers, hence n_m^2 begins with at least p ciphers, n_m^3 with at least $\frac{3}{2}p$ ciphers, *etc.*, hence the correct value of the series within the brackets as far as p places of decimals is n^m; and therefore,

$$\log(1+n_m) = \mu.n_m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Again, c being the number of places the decimal point was changed we shall evidently have

$$N = 10^{\pm c}.b\ (1+n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

where the upper sign is to be used when the decimal point is removed to the left, and the lower when removed to the right. Take the logarithm of (3),

$$\therefore \log N = \pm c\log 10 + \log b + \log(1+n) \ldots\ldots\ldots (4).$$

By (1, 2, 4), we have

$$\log N = \pm c\log 10 + \log b + \log\frac{1}{1-r_1} + \log\frac{1}{1-r_2} \ldots + \log\frac{1}{1-r_m} + \mu n_m$$

$$\ldots\ldots\ldots (5)$$

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Now, for Napierian logarithms we have, $\mu = 1$, and in the common system $\mu = M = \cdot 43429 \ldots .$; also $\log_{10} 10 = 1$.

Hence for Napierian logarithms,

$$\log_e N = \pm c \log_e 10 + \log_e b + \log_e \frac{1}{1 - r_1} + \ldots . + \log_e \frac{1}{1 - r_m} + n_m \ldots (A)$$

Also, for common logarithms,

$$\log_{10} N = \pm c + \log_{10} b + \log_{10} \frac{1}{1 - r_1} + \log_{10} \frac{1}{1 - r_2} \ldots + \log_{10} \frac{1}{1 - r_m} + M\, n_m \ldots (B).$$

Now the multiples of $\log_e 10$ are given in Table II, and $M\, n_m$ is easily calculated by means of the multiples of M in Table I; also $\log_e b$ or $\log_{10} b$ will be found in Table III, and the logarithms of $\frac{1}{1 - r_1}, \frac{1}{1 - r_2} \ldots . \frac{1}{1 - r_m}$ are given in Table IV. for both the Napierian and common systems. Hence the method of finding either the Napierian or common logarithm of any number is obvious.

5. The following will show how it is intended to arrange the work for determining the quotients $r_1, r_2 \ldots . r_m$ and last remainder n_m; and it will thus enable us to avoid explanation in the two illustrative examples of next article.

$$\begin{array}{l} 1 + n \quad (r_1 \\ \quad \underline{r_1(1 + n)} \\ \quad 1 + n_1 \quad (r_2 \\ \quad \underline{r_2(1 + n_1)} \\ \quad 1 + n_2 \quad (r_3 \\ \quad \vdots \\ \quad \vdots \\ \quad \underline{1 + n_{m-1}} (r_m \\ \quad\quad r_m (1 + n_{m-1}) \\ \quad \underline{1 + n_m} \end{array}$$

6. Example I. Required the Napierian logarithm of 665·8 to about 10 places of decimals.

$$\begin{array}{l} 5 \,|\, 6{\cdot}658 \\ \quad \underline{1{\cdot}3316} \,(\cdot 2 = r_1 \\ \quad\quad 26632 \\ \quad \underline{1{\cdot}06528} \,(\cdot 06 = r_2 \\ \quad\quad 639168 \\ \quad \underline{1{\cdot}0013632} \,(\cdot 001 = r_3 \\ \quad\quad 1001\,3632 \\ \quad \underline{1{\cdot}000361\,8 \,|\, 368} (\cdot 0003 = r_4 \\ \quad\quad 300\,1 \,|\, 0855 \\ \quad \underline{1{\cdot}000061 \,|\, 7\,2825} \,(\cdot 00006 = r_5 \text{ or } r_m \\ \quad\quad 60 \,|\, 0\,0370 \\ \quad \underline{1{\cdot}000001\,7\,2455} = 1 + n_m \end{array}$$

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Hence, recollecting that $c = 2$, and $b = 5$, we have by (A),

$$\begin{aligned}
+ 2 \log 10 &= 4{\cdot}60517018599 \\
\log 5 &= 1{\cdot}60943791243 \\
-\log(1-{\cdot}2) &= \quad 22314355131 \\
-\log(1-{\cdot}06) &= \quad 6187540372 \\
-\log(1-{\cdot}001) &= \quad 100050033 \\
-\log(1-{\cdot}0003) &= \quad 30004501 \\
-\log(1-{\cdot}00006) &= \quad 6000180 \\
n_m &= \quad 172455 \\
\log 665{\cdot}8 &= 6{\cdot}50098932514
\end{aligned}$$

N. B. Here, for convenience, the subscript e is omitted, and $-\log(1-r)$ is written for its equal $\log \frac{1}{1-r}$.

Example II. Required the common logarithm of ·00273944 to 7 or 8 decimals.

$$\begin{array}{r|l}
2 & 2{\cdot}73944 \\
\hline
 & 1{\cdot}36972 \;(\cdot 2 \\
 & \quad 27394\,4 \\
\hline
 & 1{\cdot}09577\,6 (\cdot 08 \\
 & \quad 8766\,208 \\
\hline
 & 1{\cdot}00811\,392\;(\cdot 008 \\
 & \quad 806\,491 \\
\hline
 & 1{\cdot}00004\,901
\end{array}$$

$$\begin{aligned}
-c &= -3 \\
\log 2 &= \quad {\cdot}301029996 \\
-\log(1-{\cdot}2) &= \quad 96910013 \\
-\log(1-{\cdot}08) &= \quad 36212173 \\
-\log(1-{\cdot}008) &= \quad 3488328 \\
M n_m = \begin{cases} M\times{\cdot}00004 \\ M\times{\cdot}000009 \\ M\times{\cdot}0000001 \end{cases} &= \quad 17372 \\
 &= \quad 3909 \\
 &= \quad 4 \\
\log{\cdot}00273944 &= \overline{3}{\cdot}437661795
\end{aligned}$$

We shall now consider the converse problem, namely, to find the number corresponding to a given logarithm, that is to determine the anti-logarithm of a given number.

7. Let it be required to find the anti-logarithm of any number N to p places of decimals.

Reduce N, by means of Tables II. and III, to a number between 0 and $-\log 2$, by taking from it, if it be positive, the least multiple of log 10 that exceeds it, or, if it be negative, by adding to it the greatest multiple of log 10 that is numerically less than it, and to the result add the logarithm of the greatest digit (b) that allows the sum ($-n$) to be negative.

Take, from Table IV. $\log \frac{1}{1-r_1}$, the nearest logarithm less than n, and let $-n + \log \frac{1}{1-r_1} = -n_1$; take $\log \frac{1}{1-r_2}$ the nearest logarithm less than n_1, and let $-n_1 + \log \frac{1}{1-r_2} = -n_2$, repeat this process until the sum ($-n_m$) divided by the modulus, begins with at least $\frac{1}{2}\,p$ ciphers.

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It is obvious that

$$-n = -\log\frac{1}{1-r_1} - n_1 = \log(1-r_1) - n_1.$$
$$-n_1 = -\log\frac{1}{1-r_2} - n_2 = \log(1-r_2) - n_2.$$
$$\vdots$$
$$-n_{m-1} = -\log\frac{1}{1-r_m} - n_m = \log(1-r_m) - n_m.$$

Add these equations and reject the common terms,

$$\therefore -n = \log(1-r_1) + \log(1-r_2) \ldots\ldots\ldots + \log(1-r_m) - n_m$$

Hence if a be the base, and μ the modulus, we have

$$a^{-n} = (1-r_1)(1-r_2)\ldots\ldots\ldots(1-r_m).\, a^{-n_m} \quad \ldots\ldots\ldots(6)$$

Moreover, by the exponential theorem,

$$a^{-n_m} = 1 - \frac{n_m}{\mu} + \tfrac{1}{2}.\left(\frac{n_m}{\mu}\right)^2 - etc.$$

Now $\frac{n_m}{\mu}$ is a decimal beginning with at least $\frac{1}{2}\,p$ ciphers, therefore $\left(\frac{n_m}{\mu}\right)^2$ begins with at least p ciphers, *etc.*; hence the correct value of a^{-n_m} as far as p places is $1 - \frac{n_m}{\mu}$.

$$\therefore a^{-n_m} = 1 - \frac{n_m}{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Again, c being the multiple of log 10, added or subtracted, we have,

$$N = \pm c \log 10 - \log b - n;$$

where the upper or lower sign is to be taken according as N is positive or negative,

$$\therefore a^{N} = 10^{\pm c}.\frac{1}{b}.\, a^{-n} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Hence, by (6, 7, 8) we have,

$$\text{Antilog. } N = a^{N} = 10^{\pm c}.\frac{1}{b}.(1-r_1)(1-r_2)..(1-r_m)\left(1-\frac{n_m}{\mu}\right)..(9)$$

Now, if $a = e$, $\mu = 1$, and if $a = 10$, $\mu = M$;

Hence for the Napierian system,

$$\text{Antilog. } N = e^{N} = 10^{\pm c}.\frac{1}{b}\,(1-r_1)(1-r_2)..(1-r_m)\,(1-n_m)....(C)$$

and, for the common system,

$$\text{Antilog. } N = 10^{N} = 10^{\pm c}.\frac{1}{b}.(1-r_1)(1-r_2)..(1-r_m)\left(1-\frac{n_m}{M}\right)..(D)$$

Of this last formula it may be observed that $\frac{n_m}{M}$ is easily calculated by Table II.

8. Let $1 - n_m$ or $1 - \frac{n_m}{M}$, (as the case may be) $= P_m$

$$(1-r_m)\,P_m = P_m - r_m P_m = P_{m-1}$$
$$(1-r_{m-1})\,P_{m-1} = P_{m-1} - r_{m-1}P_{m-1} = P_{m-2}$$
$$\vdots$$
$$(1-r_1)\,P_1 = P_1 - r_1 P_1 = P$$

and

$$P \div b = Q$$

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Then will antilog $N = 10^{\pm c}$. Q; hence, having found Q, we shall obtain the anti-logarithm of N, by removing the decimal point c places to the right or left according as N is positive or negative. The calculation of Q may be arranged as follows:—

$$
\begin{array}{l}
\quad P_m \quad (r_m \\
\hline
r_m \; P_m \\
\hline
\quad P_{m-1} \quad (r_{m-1} \\
\hline
r_{m-1} \; P_{m-1} \\
\hline
\quad P_{m-2} \quad (r_{m-2} \\
\quad \vdots \\
\hline
\quad P_1 \quad (r_1 \\
\hline
r_1 \; P_1 \\
\hline
b\,[\; P \\
\hline
\quad Q
\end{array}
$$

9. Example I. Required the number whose logarithm, in the common system is 2·7332059509.

$$
\begin{array}{rl}
2{\cdot}7332059509 & \\
3 & = 3 \log 10 \\
\hline
-\ {\cdot}2667940491 & \\
{\cdot}2218487496 & = -\log (1 - {\cdot}4) \\
\hline
-\ {\cdot}0449452995 & \\
409586077 & = -\log (1 - {\cdot}09) \\
\hline
-\ {\cdot}0039866918 & \\
39263455 & = -\log (1 - {\cdot}009) \\
\hline
-\ {\cdot}0000603463 & \\
434316 & = -\log (1 - {\cdot}0001) \\
\hline
-\ {\cdot}0000169147 & \\
130290 & = -\log (1 - {\cdot}00003) \\
\hline
-\ {\cdot}0000038857 & = -n_m
\end{array}
$$

$$
\frac{n_m}{M} = \left\{
\begin{array}{r}
0000069078 \\
18421 \\
1842 \\
115 \\
16 \\
\hline
-\ {\cdot}0000089472
\end{array}
\right.
$$

$$
\begin{array}{l}
1{\cdot} \\
\hline
{\cdot}99999 \,|\, 1\ 0\ 5\ 28 \,(\cdot 00003 \\
\qquad 2 \,|\, 9\ 9\ 9\ 97 \\
\hline
{\cdot}99996\ 1 \,|\, 0\ 5\ 31 \,(\cdot 0001 \\
\qquad 9\ 9 \,|\, 9\ 9\ 61 \\
\hline
{\cdot}999861\ 0 \,|\, 5\ 70 \,(\cdot 009 \\
\qquad 8998\ 7 \,|\, 4\ 95 \\
\hline
{\cdot}9908623\ 0 \,|\, 75 \,(\cdot 09 \\
\qquad 891776\ 0 \,|\, 77 \\
\hline
{\cdot}9016846\ 9\ 98 \,(\cdot 4 \\
\qquad 3606738\ 7\ 99 \\
\hline
{\cdot}5410108\ 1\ 99
\end{array}
$$

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Hence removing the decimal point three places to the right, the number is 541·0108199.

Example II. Required the number whose Napierian logarithm is —4·2520515563.

$$
\begin{array}{rl}
-4\cdot2520515563 & \\
2\cdot3025850930 & = \log 10 \\
\hline
-1\cdot9494664633 & \\
1\cdot9459101491 & = \log 7 \\
\hline
-\ \cdot0035563142 & \\
30045090 & = -\log(1-.003) \\
\hline
-\ \cdot0005518052 & \\
5001250 & = -\log(1-.0005) \\
\hline
-\ \cdot0000516802 & \\
500013 & = -\log(1-\cdot00005) \\
\hline
-\ \cdot0000016789 & \\
1\cdot & \\
\hline
\cdot99999\,8\,3211 & (\cdot00005 \\
4\,|\,9\,9999 & \\
\hline
\cdot99994\,8\,|\,3\,212 & (\cdot0005 \\
499\,|\,9\,742 & \\
\hline
\cdot999448\,3\,|\,470 & (\cdot003 \\
2998\,3\,|\,450 & \\
\hline
7)\cdot9964500\,020 & \\
\cdot1423500\,003 &
\end{array}
$$

Hence removing the decimal point one place to the left, the required number is ·01423500043.

Scholium. The methods which have now been explained, were discovered nearly seven years ago. The process of art. 4 was suggested by perusing, in the work alluded to in art. 2, Mr. Manning's ingenious method of computing logarithms (of which it may be accounted a modification), and this led me to consider the converse problem. It will be observed that in each case, the antilogarithm is resolved into factors. This mode of development may frequently be effectually employed in approximating to the value of an unknown quantity; as an example, I may perhaps be allowed to refer to my "New Method of solving Numerical Equations of all Orders." There are many other equations besides the common algebraical ones, to which the same method of continuous approximation applies with advantage. I shall probably solicit the insertion of some instances in a future number of "the Mathematician."

Newcastle-upon-Tyne,
August 18*th*, 1845.

*** The Editors take this opportunity of cordially recommending to the notice of mathematical students, Mr. Weddle's ingenious tract—"A New, Simple, and General Method of Solving Numerical Equations of all Orders." The method consists in a succession of transformations, the root of the original equation being obtained in the form of a continued product, and it is particularly adapted to the resolution of equations having several absent terms. The solution of an equation of the 20th degree with all its terms is given in the work, and also the solution of one of the 622nd degree, having only four terms.

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ON CONFOCAL CONIC SECTIONS.

[*Professor Hearn, Royal Military College, Sandhurst.*]

The following mode of investigating some properties of confocal conic sections, may perhaps interest some of the readers of the Mathematician.

Let an ellipse and hyperbola intersect in four points A, B, C, D, and let them have the same focus F. Let EP be the director of the ellipse and HP that of the hyperbola, P being the point of intersection of these directors, and FE perpendicular to EP. Draw FA, FB, FC, FD, and let

FA produced meet the ellipse again in Q and the hyperbola in S;
FB .....O..... R;
FC .....L..... M;
FD .....K..... N.

Then the lines AB, CD, MN, OQ, KL, RS being drawn will all tend to the point P, the intersection of the directors.

Let $u = ax + by + c = 0$, be the equation of EP,
$v = a'x + b'y + c' = 0$,.....HP,
and $r = \sqrt{(x^2 + y^2)}$: the focus being the origin of rectangular co-ordinates.

Then since the length of a perpendicular on EP from a point (xy) in the ellipse is $(ax + by + c)(a^2 + b^2)^{-\frac{1}{2}}$, and a perpendicular on HP from a point (xy) in the hyperbola, $(a'x + b'y + c')(a'^2 + b'^2)^{-\frac{1}{2}}$, we may denote these curves thus:—

$r = u$, the equation of the ellipse;
$r = v$, ... branch of the hyperbola through A, B;
$-r = v$, ... C, D.

Eliminating r, we have

$u - v = 0$, for the equation of AB,
$u + v = 0$,.....CD.

Also from $r = u$, and $r = v$, we get

$$(c' - c)\, r = c'u - cv \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\alpha)$$

which being squared is an homogeneous equation of the second degree in x and y, and represents two straight lines through the origin, viz. FA and FB.

Changing the sign of r in (α) we get

$$-(c' - c)\, r = c'u - cv \ldots\ldots\ldots\ldots\ldots (\beta),$$

which represents FO and FQ, the prolongations of FB and FA. And making $r = u$ in (β), we have

$$-(c' - c)\, u = c'u - cv, \text{ or } (2c' - c)\, u - cv = 0,$$

for the equation of QO.

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Again, from $r = u$, and $-r = v$, we have

$$(c' + c)\, r = c'u - cv, \text{ for F C, F D; and}$$
$$-(c' + c)\, r = c'u - cv, \text{ for F K, K L;}$$

$\therefore -(c' + c)\, u = c'u - c\, v$, or $(2\, c' + c)\, u - cv = 0$, is the equation to K L.

Similarly, $(2\, c - c')\, v - c'\, u = 0$, and $(2\, c + c')\, v - c'\, u = 0$, are the equations to R S and N M.

Now these equations to AB, CD, MN, OQ, KL, RS, are satisfied by $u = 0$, $v = 0$, which refer to the point P, the intersection of the directors, and therefore all these lines pass through the point P.

Again,—If a quadrilateral be inscribed in an ellipse such that its diagonals shall pass through one of the foci, the pairs of opposite sides being produced shall intersect in the director; and if the points of intersection be joined with the focus, the joining lines shall bisect the angles between the diagonals.

This is only a particular case of the former corresponding to $c' = 0$, the hyperbola becoming the two straight lines AD, CB. But I propose to give an a priori proof.

Let F, the focus, be the origin, FV bisecting the angle CFD, the axis of x, and put angle CFV $= a$ (ABDC being the inscribed quadrilateral).

Then

$r \cos a = x$, is the equation to CF and FD,

$r = u$, is the equation to the ellipse, where $u = 0$, represents EP the director, as in the former part.

Hence $u \cos a = x$, is the equation to CD.

Again, $-r \cos a = x$, represents FA and FB;

$\therefore u \cos a = -x$, is the equation to AB.

The equations of AB and CD are satisfied by $u = 0$, $x = 0$, and therefore they intersect in EP and a line at right angles to FV through F.

Also $r \sin a = y$, is the equation to FC, FA;

$-r \sin a = y$ ..... FB, FD;

$\therefore u \sin a = y$, and $u \sin a = -y$, are the equations to AC and BD which are satisfied by $u = 0$, $y = 0$, and therefore the intersection of CA, DB is also that of PE, VF.

The well known theorem that if a surface of the second order (of revolution) be cut by a plane, the curve of section viewed from the focus will appear circular, may be thus briefly demonstrated.

Let $r = ax + by + cz + d$, be the surface,

and $x = m$, the plane of section.

Then $r = \left(a + \frac{d}{m}\right) x + by + cz$, is the equation to a surface through the curve of section and the origin or focus; and this is the equation to a right cone, since

$$r = \sqrt{(x^2 + y^2 + z^2)}.$$

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ON THE GENERAL REDUCTION OF CERTAIN INTEGRALS.

[*Professor F. W. Newman, New College, Manchester.*]

I. GENERAL REDUCTION OF $\int Fx \log (ax+b)dx$, WHEN F IS RATIONAL.

1. We do not limit the problem by supposing $a=1$, $b=0$; for by assuming $ax+b=x'$, and $F(x)=F_1(x')$, F_1 is rational as F was, so that the integral is reduced to the form $\int Fx \log x\, dx$.

2. Separate from Fx its integer part, as in dealing with $\int Fx\, dx$; and we have to find in the first place $\int(\Sigma . Ax^n) \log x\, dx$, or $\Sigma . A\int x^n \log x\, dx$. But

$$\int x^n \log x\, dx = \frac{x^{n+1}}{n+1}\left\{\log x - \frac{1}{n+1}\right\};$$

so that the integer part of Fx involves no difficulty.

3. Let the remainder of Fx be denoted by $\Sigma\left(\frac{C}{x-c}\right)+\Sigma\left(\frac{C'}{(x-c)^m}\right)$, where c is real or imaginary, and m is a positive integer greater than unity. We may then entirely integrate all that depends on m, as follows:—

$$\text{Let } U_{n-1} = \int \frac{x \log x}{(x-c)^{n-1}}, \text{ and } V_m = \int \frac{\log x\, dx}{(x-c)^m};$$

then by differentiating U_{n-1}, and again integrating, we get

$$\begin{aligned}
&\int -\frac{dx}{(x-c)^{n-1}} - U_{n-1} \\
&\quad = (n-1)\int \frac{x \log x\, dx}{(x-c)^n} - \int \frac{\log x\, dx}{(x-c)^{n-1}} \\
&\quad = (n-1)\int \frac{(x-c)\log x\, dx}{(x-c)^n} + (n-1)\, c \int \frac{\log x\, dx}{(x-c)^n} - \int \frac{\log x\, dx}{(x-c)^{n-1}} \\
&\quad = (n-1)\, c\, V_n + (n-2)\, V_{n-1}.
\end{aligned}$$

Hence we get the equation of reduction,

$$(n-1)\, c\, V_n + (n-2)\, V_{n-1} = \int \frac{dx}{(x-c)^{n-1}} - U_{n-1};$$

by help of which V_m is successively reduced to $V_{m-1}, V_{m-2} \ldots\ldots V_3, V_2$.

Make $n=2$ in the last equation and we get (if c is real)

$$c\, V_2 = \int \frac{dx}{x-c} - U_1 = \log (x-c) - \frac{x \log x}{x-c}.$$

If c is imaginary, F having been a real function, there are two corresponding roots of the form $c = \alpha + \beta\sqrt{-1}$, $c' = \alpha - \beta\sqrt{-1}$; and combining them as the case may require, we have an integral of the form

$$A\int \frac{dx}{x-c} + A'\int \frac{dx}{x-c'};$$

where A and A′ are corresponding imaginaries, like c and c'. This sum is evidently real, and can be integrated by logarithms and circular arcs. Hence in all cases V_2 is obtained without involving any new functions. The same then is true of $V_3, V_4, \ldots . V_m$.

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4. It only remains to find $\int \Sigma \left(\frac{C}{x-c}\right) \log x \, dx$; or a sum of terms of which each takes the form $\int \frac{\log x \, dx}{x-c}$. And *first*, let c be real. If c be negative let $c = -e$, therefore

$$\int \frac{\log x \, dx}{x+e} = \log x \log (x+e) - \int \frac{dx}{x} \log (x+e).$$

Let $x+e=y$, and the last integral $= \int \frac{\log y \, dy}{y-e}$, where e is positive. This is the same form as $\int \frac{\log x \, dx}{x-c}$, where c is positive, to which consequently the other case is reduced.

When c is positive, let $x = cz$; therefore $\log x = \log c + \log z$; whence

$$\int \frac{\log x \, dx}{x-c} = \log c \log (z-1) + \int \frac{\log z \, dz}{z-1}; \text{ if } z \rhd 1;$$

$$\text{or} = \log c \log (1-z) + \int \frac{\log z \, dz}{z-1}; \text{ if } z \lhd 1.$$

Here we have a new elementary integral, which is the first in the series of *Spence's Transcendents*. The best definition of these transcendents is the following, which excludes all objection drawn from the divergence of series.

Let Lx represent $\int_1 \frac{dx}{x}$ or $\log x$: also let $L^2(x) = \int_1 \frac{Lx \, dx}{x-1}$;

$L^3(x) = \int_1 \frac{L^2 x \, dx}{x-1}$; and generally $L^n(x) = \int_1 \frac{L^{n-1} x \, dx}{x-1}$; then L^2x, L^3x, ... L^nx, are the functions known as Spence's Logarithmic Transcendents.

Since Spence has fully discussed the properties of these, no more need here be said about them. But as his work is rather scarce and expensive, it may not be amiss here to set down the principal equations which concern L^2, from which its value is calculated. The student will easily verify them for himself.

$$L^2(1+x) = \frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots. \text{ (when } x \lhd 1).$$

$$L^2(1-x^2) = 2L^2(1+x) + 2L^2(1-x).$$

$$L^2 x + L^2(1-x) = \log x \log (1-x) + L^2(0).$$

$$L^2(1+x) + L^2\left(1+\frac{1}{x}\right) = \tfrac{1}{2}(\log x)^2 + 2L^2(2).$$

$$L^2 x - L^2\left(1-\frac{1}{x}\right) = \log x \log (x-1) - \tfrac{1}{2}(\log x)^2 - L^2(0).$$

$\therefore L^2 x = \frac{1}{2}(\log x)^2 - L^2(0)$, when $x = \infty$.

$$L^2(1+x)^2 = 2L^2(1+x) - 2L^2(2+x) + 2\log(1+x)\log(2+x) + 2L^2(2).$$

Since $L^2(0) = -1 - 2^{-2} - 3^{-2} - 4^{-2} - 5^{-2} - \ldots$, we have $L^2(0) = -\frac{\pi^2}{6}$.

Also $L^2(2) = 1 - 2^{-2} + 3^{-2} - 4^{-2} + 5^{-2} - \ldots$, or $L^2(2) = \frac{\pi^2}{12}$.

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As L^2x involves no arbitrary constant, it is suitable for tabulation. Spence has actually tabulated it, from $x = 1$ to $x = 100$; and as the equations just given enable us to find L^2x for *all* values, by help of the table, we regard L^2 as being *now* an integral known and available for further researches, like $\log x$ or $\cos x$.

5. We have to return to the case in which c is imaginary. Then, if c and c', C and C′ are corresponding imaginaries, $\left\{\frac{C}{x-c}+\frac{C'}{x-c'}\right\}$ will take as usual the form

$\frac{p(x-\cos a)+q}{x^2-2x\cos a+1}$; where a is an arc between 0 and π. Thus we are brought to two elementary integrals,

$$\phi(x,a)=\int_0 \frac{\log x\,dx}{x^2-2x\cos a+1}:\ \phi_1(x,a)=\int_0 \frac{(x-\cos a)\log x\,dx}{x^2-2x\cos a+1};$$ the resolution of which would complete our original problem.

It is easy to reduce each of them to the case of x less than unity. For observing that

$$\frac{1}{x^2}-\frac{2}{x}\cos a+1=\frac{1}{x^2}(1-2x\cos a+x^2),$$

and $$\log\left(\frac{1}{x}\right)d\left(\frac{1}{x}\right)=\frac{1}{x^2}\log x\,dx;$$

we get $$\frac{d}{dx}\phi\left(\frac{1}{x},a\right)=\frac{d}{dx}\phi(x,a).$$

Integrate, and correct by making $x=1=\frac{1}{x}$, and we get

$$\phi\left(\frac{1}{x},a\right)=\phi(x,a)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

To explain this, it must be observed that $\log x$, and therefore $\phi(x,a)$ is negative while $x \lhd 1$. When x passes 1, the new increments are positive.

Again,

$$\frac{\frac{1}{x}-\cos a}{1-2x\cos a+x^2}=\frac{1}{x}-\frac{x-\cos a}{x^2-2x\cos a+1}.$$

Multiply by $\log x\,dx$, and integrate; therefore

$$\phi_1\left(\frac{1}{x},a\right)=\tfrac{1}{2}(\log x)^2-\phi_1(x,a)+2\phi_1(1,a)\ldots\ldots\ldots(B)$$

Equations (A) and (B) reduce the case of $x \rhd 1$ to that of $x \lhd 1$.

6. When $x \lhd 1$, we may always find ϕ and ϕ_1 in series, though the convergence will not be practically available unless $x \lhd \frac{1}{2}$. Integrating by parts, we have

$$\left.\begin{aligned}\sin a.\ \phi(x,a)&=\log x\tan^{-1}\left(\frac{x\sin a}{1-x\cos a}\right)-\int_0\frac{dx}{x}\tan^{-1}\left(\frac{x\sin a}{1-x\cos a}\right)\\ \phi_1(x,a)&=\log x\log(1-2x\cos a+x^2)^{\frac{1}{2}}-\int_0\frac{dx}{x}\log(1-2x\cos a+x^2)^{\frac{1}{2}}\end{aligned}\right\}.(C)$$

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The two integrals thus resulting are obtained by expanding the tan^{-1} and *log* which they contain; namely in the forms (including their sign)

$$\left.\begin{array}{l} -x\sin a - x^2.2^{-2}\sin 2a - x^3.3^{-2}\sin 3a - x^4.4^{-2}\sin 4a - \ldots \\ +x\cos a + x^2.2^{-2}\cos 2a + x^3.3^{-2}\cos 3a + x^4.4^{-2}\cos 4a + \ldots \end{array}\right\} \ldots\ldots\ldots(D)$$

But these series cannot be employed when the value of x approaches to unity.

7. The integrals ϕ and ϕ_1 are ill suited to tabulation, because of their containing an arbitrary constant a. But we may reduce ϕ to functions of one variable, as follows:—

Let $\tan\omega = \dfrac{x\sin a}{1-x\cos a}$, and ω begin with x from 0; therefore

$$x = \frac{\tan\omega}{\sin a + \tan\omega\cos a} = \frac{\sin\omega}{\sin(\omega+a)};$$

hence, $\sin a\,\phi(x, a) = \int_0 \log x\,d\omega = \int_0 \{\log\sin\omega - \log\sin(\omega+a)\}\,d\omega$.

Let $\Psi\omega = -\int_0 \log\sin\omega\,d\omega$; hence

$$\sin a\,\phi(x, a) = \Psi(\omega+a) - \Psi\omega - \Psi a \ldots\ldots\ldots\ldots(E)$$

ϕ is therefore reduced to the function Ψ.

No similar reduction of ϕ_1 appears practicable. Before proceeding any further to investigate the properties of ϕ_1 it seems proper to direct our attention more minutely to the new function Ψ, which is altogether suitable for tabulation.

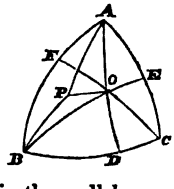
(To be continued.)

DIRECT SOLUTION OF A SPHERICAL PROBLEM.

[*The Rev. Robert Main, M. A.*]

The following investigation of the relation between the three sides of a spherical triangle, and three arcs of great circles drawn from any point of the sphere to the angular points affords a direct solution of the problem—"To find the Polar distance of a heavenly body from its measured distances from two given fixed stars."

Let α, β, γ be the sides of the triangle ABC,
a, b, c the circular arcs drawn from its angles
A, B, C, to the point O.



Let also $\theta, \theta_1, \theta_2$ be the angles subtended by the sides of the triangle at the point O; then if the point be within the triangle $\theta + \theta_1 = 360^\circ - \theta_2$; and if it be without $\theta + \theta_1 = \theta_2$.

In either case, $\cos(\theta+\theta_1) = \cos\theta_2$, from which we obtain the well known relation

$$\cos^2\theta + \cos^2\theta_1 + \cos^2\theta_2 = 1 + 2\cos\theta\cos\theta_1\cos\theta_2,$$

and replacing $\cos\theta$, $\cos\theta_1$, $\cos\theta_2$ by their values in terms of the sides of the triangles into which the original triangle is subdivided:

$$\frac{(\cos\gamma - \cos a\cos b)^2}{\sin^2 a\sin^2 b} + \frac{(\cos\beta - \cos a\cos c)^2}{\sin^2 a\sin^2 c} + \frac{(\cos a - \cos b\cos c)^2}{\sin^2 b\sin^2 c}$$
$$= 1 + 2\frac{(\cos\gamma - \cos a\cos b)(\cos\beta - \cos a\cos c)(\cos a - \cos b\cos c)}{\sin^2 a\sin^2 b\sin^2 c},$$

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or $\sin^2 c\,(\cos\gamma - \cos a \cos b)^2 + \sin^2 b(\cos\beta - \cos a\cos c)^2 + \sin^2 a(\cos a - \cos b\cos c)^2$
$= \sin^2 a \sin^2 b \sin^2 c + 2(\cos\gamma - \cos a\cos b)(\cos\beta - \cos a\cos c)(\cos a - \cos b\cos c)$.

From this equation, by a series of transformations which are sufficiently obvious, we get the following equation of remarkable symmetry,

$$\left.\begin{matrix}\sin^2\gamma\sin^2 c' \\ +\sin^2\beta\sin^2 b + 2 \\ +\sin^2 a \sin^2 a\end{matrix}\right\{\begin{matrix}\cos\gamma\cos a(\cos b - \cos c\cos a) \\ +\cos\beta\cos a(\cos c - \cos b\cos a) \\ +\cos b\cos c(\cos a - \cos\beta\cos\gamma)\end{matrix}\left.\right\} + 2\cos a\cos\beta\cos\gamma = 2.$$

Greenwich, Sept. 1st, 1845.

MATHEMATICAL NOTE.

The homogeneous equation to a plane deduced by a simple and direct process.

Take oblique co-ordinates for generality, and let AC in the plane of xz be the straight line to which the generating straight line is always parallel; BC in the plane of yz that along which it moves, A, B, C, being points in the axes. Also, let the generating line MN in any position meet CB, BA, in M and N; from which draw MQ, NQ, in the planes yz, xy, parallel, respectively, to CO, and OA, O being the origin. These lines evidently meet each other in the axis of y.

Put $AO = a$, $BO = b$, $CO = c$, and denote any point P in MN by xyz.

Then $\frac{z}{MQ} + \frac{x}{QN} = 1$..(1), $\frac{MQ}{c} + \frac{y}{b} = 1$..(2), $\frac{NQ}{a} + \frac{y}{b} = 1$....(3)

Hence by (1)

$$1 = \frac{z}{c}\cdot\frac{c}{MQ} + \frac{x}{a}\cdot\frac{a}{QN}$$
$$= \left(\frac{z}{c} + \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)^{-1}, \text{ by (2, 3)};$$

or
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

a relation which holds between the co-ordinates of any point in the generating line in any position, and hence it is the equation to the plane generated by the straight line MN.

γ.

HORNER ON ALGEBRAIC TRANSFORMATION.

(Continued from vol. 1, page 316.)

33. We may now return to the partial theorem of art. (23), which we have hitherto employed only to detect the limits of the roots of an ordinary equation. My readers will have perceived that its range is far more extensive, since it contains in a very explicit form the principle of the *calculus of finite differences*. In fact, if we take any one of the given, or of the absolute terms in that formula, and trace its connection with the rest, we

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shall be conducted through a uniform course of dependences of this kind, taking M_o as a sample of the rest;

$$\left.\begin{array}{l} M_1 - M_o = L_1, \\ L_2 - L_1 = 2K_2, \\ K_3 - K_2 = 3H_3, \\ \ldots\ldots\ldots\ldots \end{array}\right\} \text{wherefore,} \left\{\begin{array}{l} L_1 = \Delta M_o, \\ 2K_2 = \Delta L_1 = \Delta^2 M_o \\ 3H_3 = \Delta K_2 = \tfrac{1}{2}\Delta^3 M_o \\ \ldots\ldots\ldots\ldots \end{array}\right.$$

$$\text{or, } L_1 = \Delta M_o,\ K_2 = \tfrac{1}{2}\Delta^2 M_o,\ H_3 = \frac{1}{2.3}\Delta^3 M_o,\ \textit{etc.}\ *$$

A similar chain of relations connects each diagonal line of the results, that is traced to the left downwards, through the formula in art. (23). That formula may, therefore, be thus interpreted:—

……..I_o	H_o	K_o	L_o	M_o	N_o
1^n ……ΔH_o	ΔK_o	ΔL_o	ΔM_o	ΔN_o	N_1
2^{n-1} …. $\frac{1}{2}\Delta^2 K_o$	$\frac{1}{2}\Delta^2 L_o$	$\frac{1}{2}\Delta^2 M_o$	$\frac{1}{2}\Delta^2 N_o$	ΔN_1	N_2
3^{n-2} …. $\frac{1}{6}\Delta^3 L_o$	$\frac{1}{6}\Delta^3 M_o$	$\frac{1}{6}\Delta^3 N_o$	$\frac{1}{2}\Delta^2 N_1$	ΔN_2	N_3
4^{n-3} …. $\frac{1}{24}\Delta^4 M_o$	$\frac{1}{24}\Delta^4 N_o$	$\frac{1}{6}\Delta^3 N_1$	$\frac{1}{2}\Delta^2 N_2$	ΔN_3	N_4
etc.	*etc.*		*etc.*		

Where every line contains the coefficients of a transformed value of fx or N_x. The general form, then, when $m \lessdot n$, is

$$N_x = N_m + \Delta N_{m-1}\, x_m + \frac{1}{1.2}\Delta^2 N_{m-2}\, x_m\, x_{m-1} + \ldots$$
$$+ \frac{1}{1.2\ldots m}\Delta^m N_o\, x_m\, x_{m-1}\ldots x_1 + \frac{1}{1\ldots m}\Delta^m M_o\, x_m \ldots x_1 x +$$
$$\frac{1}{1\ldots m}\Delta^m L_o\, x_m \ldots x_1 x^2 + \ldots \text{ to } x^{n-m} \qquad [\text{I}]$$

And when $m = n$ or $m \gtrdot n$, the equation is simply

$$N_x = N_m + \Delta N_{m-1}\, x_m + \frac{1}{1.2}\Delta^2 N_{m-2}\, x_m\, x_{m-1} + \ldots$$
$$+ \frac{1}{1\ldots n}\Delta^n N_{m-n}\, x_m \ldots x_{m-n+1} \qquad [\text{II}]$$

34. As the coefficients of the latter of these two formula consist entirely of the absolute terms and their differences, it disengages us from any consideration relative to the algebraic formula or *terminus generalis* of which N is a value, and supplies directions relative to the *summation and interpolation of series.*

This formula differs from that which is now in use, and is new to me. Possessing every advantage of the system of continuous approximation, and harmonising with the other parts of that system, it seems to merit a particular elucidation.

* The reader will observe that in this and some following articles a subordinate coefficient with the differential coefficient prefixed, is regarded for the time as the primitive or absolute term relative to all that are inferior to it; whereas in art (22) every term is considered in relation to the real absolute term. The equation between the two values is easily deduced from one formula.

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35. In the first place, therefore, let us examine the relation which it bears to the rule in art. (24) for reducing it to a pure transformée. And here, adopting the notation DN, D^2N, *etc.*, to designate the ordinal relation of a set of such reduced differences to their absolute term N, we perceive that the rule in question may be thus tabulated, commencing with the m^{th} line of the preceding table, or its prototype in art. (23).

....:	I_m	H_m	K_m	L_m	M_m	N_m
m^{n-m} or $(n-1)^1$....	DH_m	DK_m	DL_m	DM_m	DN_m	
m^{n-m-1} or $(n-2)^1$....	D^2K_m	D^2L_m	D^2M_m	D^2N_m		
m^{n-m-2} or $(n-3)^1$....	D^3L_m	D^3M_m	D^3N_m			
⋮	⋮	⋮				
.....:	⋮	⋮				

The marginal directions for multiplying (art. 23) are adapted to the cases of $m \lhd n$, and $m = n$ or $m \rhd n$.

36. But we may observe that this notation is not merely expressive of ordinal, but of derivative dependence. To understand the nature of the operations which it injoins, we must previously take a more enlarged view of the differential notation, by considering what would have been the consequence of making the constant increment of the variable in art. (23) indefinite, or a instead of 1. And we immediately perceive that the connecting multipliers would have been a times what they are in arts. (23) and (33), and that we should have had

$$M_1 - M_o = a\,L;\ L_2 - L_1 = 2a\,K_2;\ \textit{etc.}: \text{— or,}$$

$$L_1 = \frac{\Delta M_o}{a} = \frac{\Delta M_o}{\Delta x};\ K_2 = \frac{\frac{1}{2}\Delta^2 M_o}{(\Delta x)^2};\ H_3 = \frac{\frac{1}{2.3}\Delta^3 M_o}{(\Delta x)^3};\ \textit{etc.}$$

37. Reserving now, as is usual, the notation by Δ to express the case when the increment is 1, and employing d to mark that of the zero-increments or infinitesimal differences, used in the rule in question, we shall have in commencing from any pure transformée,

	I_o	H_o	K_o	L_o	M_o	N_o
(0×1^n)	$\frac{dH_o}{dx}$	$\frac{dK_o}{dx}$	$\frac{dL_o}{dx}$	$\frac{dM_o}{dx}$	$\frac{dN_o}{dx}$	N_o
$(0 \times 2^{n-1})$	$\frac{\frac{1}{2}d^2K_o}{(dx)^2}$	$\frac{\frac{1}{2}d^2L_o}{(dx)^2}$	$\frac{\frac{1}{2}d^2M_o}{(dx)^2}$	$\frac{\frac{1}{2}d^2N_o}{(dx)^2}$	$\frac{dN_o}{dx}$	N_o
	etc.		*etc.*		*etc.*	

a formula in which, as all the addends are multiples of zero, every column consists of a repetition of the same identical quantity variously represented, according to the diversity of the absolute term or primitive function from which it may be supposed to originate.

In reference to N_o, the actual primitive, we have,

$$M_o \text{ or } DN_o = \frac{dN_o}{dx};\ L_o \text{ or } D^2N_o = \frac{\frac{1}{2}d^2N_o}{(dx)^2}: \text{ and generally}$$

$$D^nN_o = \frac{d^nN_o}{1\cdot 2\,..\,n(dx)^n};\ \text{so that, } N_x = N_o + DN_o\,x + D^2N_o\,x^2 + D^3N_o\,x^3 + ..$$

which is equivalent to *Taylor's theorem.*

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It is manifest that had we begun with the coefficients of the more general expression

$$N_x = N_m + DN_m\, x_m + D^2N_m\, x_m^2 + \ldots.$$

we should have found in the same manner $D^nN_m = \frac{d^nN_m}{1\cdot 2\ldots n(dx)^n}$.

Consequently the coefficients of every pure transformée are the same as the derived functions, found by taking the successive infinitesimal differences of the primitive or absolute term, when expressed in a general form. In our processes the differentiation is latent, because it accompanies substitution, instead of being made to precede it, as is the usual practice.

I do not stop to notice various points of connection between finite and infinitesimal differences, which might now be the three last tabulated formula; but hasten to the practical application of what has been said.

38. In preparing to operate upon any given series according to the principle of equation ii (art. 33) the successive orders of differences are taken as usual, until they vanish. But instead of the first, we are to select the *last* in that portion (say l terms) of the series which we employ, and in the several courses of differences derived from that portion.

If a sum, of any order, is required to be found, the successive sums of the partial series of l terms must be found as far as the required order, and all of these, except the highest order being counted among the successive differences appertaining to it, the last in each set is to be selected as before.

Let the orders of differences be n in number; their quotients by 1, 2, 6, 24, *etc.*, are to be arranged in any convenient manner in retrogade order.

The set of multipliers by which such a series must be supposed to have been derived from the next previous series in art. 23 or 33, being n, n—1, ..2, 1, and the increment which the series is yet to receive being $x-l$ terms; the multipliers for completing it are $\overline{x-l} + \overline{n-1}$ and its remainders when 1 is continually deducted.

Ex. 1. What is the cube of 74?

			0				
		1	1				
	6	7	8				
6	12	19	27	$= 3^3$	$\therefore 74-3+2=73$	= first multiplier.	
				1	6	19	27
					73	158	5707
						553	39949
				73^1.....		79	5707	405224 *Ans.*

Ex. 2. What is the amount of all the cubes from 1^3 to 74^3?

			0	0				
		1	1	1				
	6	7	8	9				
6	12	19	27	36	$\therefore 74-3+3=74$	= first multiplier.		
			$\frac{1}{4}$	2	$9\frac{1}{2}$	27	36	
				$18\frac{1}{2}$	1460	3012	108459	
					$36\frac{1}{2}$	10542	759213	
			74^1.....	$\frac{1}{4}$	$20\frac{1}{2}$	1506	108459	7700625 *Ans.*

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Ex. 3. What is the n^{th} term of the second series of 1, 4, 7,.. ?

	1	1	1			
	3	4	5	6		
0	3	7	$\therefore n-2+2=n$			
$\frac{1}{2}$		2		5		6
		$\frac{n}{2}$	$\frac{n^2}{2}+2n$		$\frac{n^3}{2}+\frac{3n^2}{2}+3n$	
			$-\frac{n}{2}-2$		$-\frac{2n^2}{2}-3n-6$	
n^1....	$\frac{1}{2}$	$\frac{n}{2}+2$	$\frac{n^2}{2}+\frac{3n}{2}+3$		$\frac{n^3}{2}+\frac{n^2}{2}=\frac{1}{2}n^2(n+1)$.	

Ans.

Ex. 4. To find the common logarithm of 3.1415926536 by means of a table containing the logarithms of numbers from 1 to 1000 to ten decimals. [*Lacroix, Eng. Ed.* 407.]

NO.	LOGS.	Δ^1	Δ^2	Δ^3	Δ^4
310	4913616938				
311	4927603890	13986952			
312	4941545940	13942050	—44902		
313	4955443376	13897436	—44614	288	
314	4969296481	13853105	—44331	283	—5

	$\frac{1}{24}\Delta^4$	$\frac{1}{6}\Delta^3$	$\frac{1}{2}\Delta^2$	Δ^1	
—	0·2	+4 7·2	—22 16 55	+ 13853 10 5·6	+ 4969296481
		—62	9 32	—22 06 7	1382883 2
		4 66	47	— 2 20 68	691276 5
3^1			—22 06 76	13828 8 3 16	124418 0
			4 66	—2 20 1	2764 8
			47	—1 10 1	829 4
			23	13825 5 30	69 1
			—22 01 40	—1 10	4 1
115^1				—20	8
				1382 4 23	4971498726 9
				1382 4 0	

39. In this example, the whole increment 159.. is introduced by one digit at a time as in arts. 27 *etc.* Wherefore, the virtual multipliers by which the line of data may be supposed to have been found, being 4, 3, 2, 1, (*art.* 23), those for the first course of operation are 31, 21, 11, 1. The next course is by the previous contractions reduced to an operation of the *third* order; consequently, of the multipliers 215, 115, 15, 5, the last three only are employed. From this period the work is of exactly the same character as that of an ordinary equation.

In fact, if the solution of the equation in *art.* 30 had been deduced immediately from the formula

1 5 16 42 99 —207

instead of being prepared by an intervening transformation, the first course of multipliers would then have been 46, 36, 26, 16, 6; and the next 363, 263, 163, 63, 3, and so on; unit-increments being for a short distance intermixed with digit-increments.

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This diversity of increments can occasion little embarrassment, if the student trust in every case, to the concise and universal precept of the *General Theorem (art.* 15.*)*, otherwise, the present example might, with little if any loss of facility, have been prepared in the manner of the equation alluded to, by a previous reduction to a pure transformée. Thus

	—2	47 2	—22165 5	13853105	496926481
	0	—6	93 2	—22072 3	
3^1	—2	46 6	—22072 3	13831032 7	
	0	—4	46 2		
2^1	—2	46 2	—22026 1		
	0	—2			
1^1	—2	46			

The mode of operating on the formula

—·2 46· —22026·1 13831032.7 496926481

would then be entirely of the character of the ulterior solution in *art.* 30; viz., by digit-increments solely: or it would suit *any mode* of continuous approximation.

It deserves to be remarked, as a means of verification, that the formula just written is equivalent to log 314 and its derivative functions *(art.* 37*)*: viz., the quotients produced by continually dividing the modulus ·4342944819 by 314, 2 (314), ½ (314), *etc.* Also, that the sums of the last column in the example, if taken at every line would produce the successive logs of 3141, 31415, 314159, *etc.*

(To be continued.)

DEMONSTRATION OF A GEOMETRICAL THEOREM.

[*Mr. John Davis, Skibbereen, County of Cork.*]

Theorem. If from any point in the plane of a triangle perpendiculars be drawn to the sides, and the extremities of these perpendiculars be joined two and two; if the triangle thus formed be denoted by Δ', the triangle by joining two and two the points of bisection of the sides of the original triangle by Δ, the radius of the circumscribed circle by R, and the distance between the assumed point and the centre of the circumscribed circle by R′; then

$$\frac{\Delta'}{\Delta} = \pm \frac{R^2 - R'^2}{R^2},$$

according as the assumed point is *within* or *without* the circumscribed circle.

Let P be the assumed point in the plane of the triangle ABC, and QRS the triangle represented by Δ'. Let O be the centre of the circumscribed circle, and AOK a diameter. Join CP and produce it to meet the circumference of the circle in D, and draw BD. Join OP and CK, and draw the perpendiculars AH, BI, PE, RF to the lines BC, CA, BD, QS respectively, and produce QP to meet BA in G. Then, since each of the quadrilaterals BSPQ and PQCR is inscriptible in a circle, we have angle DBA = DCA =

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PQR, and ABP = PQS; hence angle EBP = FQR, and the triangles EBP and FQR are similar; therefore

$$PE : RF :: PB : QR, \text{ or } PE \cdot QR = PB \cdot RF.$$

From the similar triangles BPG and GSQ we get

$$QS : BP :: GS : GP :: GQ : GB :: AH : AB :: AC : AK;$$

$$\therefore \quad QS : BP :: AC : AK, \text{ or } QS \cdot RF : BP \cdot RF :: AC : AK.$$

Similarly, $RQ : PC :: AB : AK$ .....(1)

Again; the angles EDP, BAC being equal, we have

$$EP : PD :: BI : AB .....(2)$$

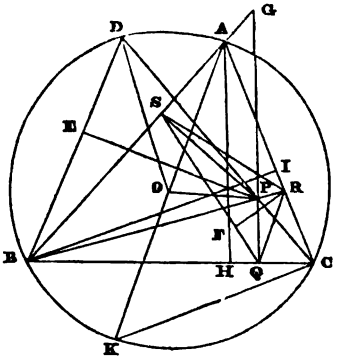
By (1) and (2),

$$RQ \cdot PE : CP \cdot PD :: AB \cdot BI : AB \cdot AK :: BI : AK$$

But it has been shown that $RQ \cdot PE = PB \cdot RF$; hence

$$PB \cdot RF : CP \cdot PD :: BI : AK,$$

and since $QS \cdot RF : PB \cdot RF :: AC : AK$;



$$\therefore \quad QS \cdot RF : CP \cdot PD :: AC \cdot BI : AK^2.$$

Now $QS \cdot RF = 2\Delta'$, $CP \cdot PD = DO^2 - OP^2 = R^2 - R'^2$, $AC \cdot BI = 8\Delta$, and $AK^2 = 4R^2$; consequently, we get finally

$$\Delta' : \Delta :: R^2 - R'^2 : R^2.$$

In a similar manner, when the point P is *without* the circumscribed circle, it may be proved that

$$\Delta' : \Delta :: R'^2 - R^2 : R^2$$

consequently, $$\frac{\Delta'}{\Delta} = \pm \frac{R^2 - R'^2}{R^2},$$

according as the point is *within* or *without* the circumscribed circle.

*** Agreeably to the note at p. 73, vol i. of the "Mathematician," the Editors have much pleasure in presenting to the notice of their readers, the preceding geometrical demonstration of a theorem, elegant in itself, and interesting in its applications. One or two of these applications are noticed in the page already quoted, and an analytical proof of the theorem itself is given at pp. 71, 72.

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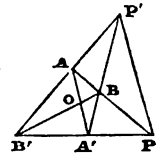
MATHEMATICAL NOTES.

[*Mr. J. W. Elliott, Greatham.*]

1. ABA'B' is a quadrilateral, of which the diagonals are AA', BB', intersecting in O, and such that AO = OA'. Let the opposite sides AB, A'B' be produced to meet in P, and A'B, AB' in P': then PP' is parallel to AA'.



Take OA', OB, for the axes of x and y respectively, and put $OA' = OA = a$, $OB = \beta$, $OB' = \beta_1$;

then the equations of the several lines are as below:

$$(A'B) \ldots\ldots\ldots \frac{x}{a} + \frac{y}{\beta} = 1 \qquad (AB) \ldots\ldots - \frac{x}{a} + \frac{y}{\beta} = 1$$

$$(AB') \ldots\ldots\ldots - \frac{x}{a} - \frac{y}{\beta_1} = 1 \qquad (A'B') \ldots\ldots\ldots \frac{x}{a} - \frac{y}{\beta_1} = 1$$

Now since A'B, AB' meet in P', and AB, A'B' in P, we have from these equations

$$\frac{y}{\beta} - \frac{y}{\beta_1} = 2, \quad \frac{y}{\beta} - \frac{y}{\beta_1} = 2,$$

which are identical. PP' is therefore parallel to AA'.

2. An inextensible string binds tightly together two smooth cylinders of given radii r, r'; to find the ratio of the mutual pressure (P) between the cylinders to the tension (T) by which it is produced. (From Walton's Mech. Prob. p. 78).

Let O, O' be the centres of the two circular sections ABC, A'B'C' respectively, and β the inclination of AO or A'O' to OO'; then

$$P = 2T \sin\beta, \text{ or } \frac{P}{T} = 2\sin\beta;$$

but since

$$\cos\beta = \frac{r - r'}{r + r'}, \text{ or } \sin\beta = \frac{2\sqrt{rr'}}{r + r'};$$

$$\therefore \quad \frac{P}{T} = \frac{4\sqrt{rr'}}{r + r'},$$

the ratio required.

The length (L) of the string may also be readily found; for let ABC, A'B'C' be the arcs of contact of the string with the cylinders; then

$$ABC = 2r\pi - 2r\beta, \quad A'B'C' = 2r'\beta,$$

$$AA'^2 = (r + r')^2 - (r - r')^2 = 4rr';$$

hence

$$L = 4\sqrt{rr'} + 2r\pi - 2(r - r')\cos^{-1}\frac{r - r'}{r + r'}.$$

When the cylinders are equal, that is, when $r = r'$; then $P = 2T$, and $L = 2r(\pi + 2)$.

September 6th, 1845.

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MATHEMATICAL EXERCISES. *(continued.)*

57. *Mr. J. W. Elliott, Greatham.*

If a straight line be drawn from the vertex of any plane triangle, to the point of contact of either the inscribed or escribed circle; then the middle points of that line and the base, and the centre of the circle, range in the same straight line.

58. *By F.*

Prove, without the aid of goniometry, the following relation which we may name the *Reciprocal Theorem:*

$$x^n + \frac{1}{x^n} = y^n - ny^{n-2} + \frac{n\,(n-3)}{1\cdot 2}\,y^{n-4} - \frac{n\,(n-4)\,(n-5)}{1\cdot 2\cdot 3}\,y^{n-6} + \frac{n\,(n-5)\,(n-6)\,(n-7)}{1\cdot 2\cdot 3\cdot 4}\,y^{n-8} - \ldots.$$

where $y = x + \frac{1}{x}$.

59. *By γ.*

If the outer envelope be removed from all the faces of a complete pile of shot, give an expression for the number of shot left in each kind of pile; and if the outer border be removed from each kind of course, give an expression for the number of shot left.

60. *Mr. J. W. Elliott, Greatham.*

Find the moment of inertia of a regular polygon of n sides revolving about an axis through its centre, and making an angle α with its plane.

61. *Mr. Thomas Weddle, Newcastle.*

1. Let ABC, A_1BC, AB_2C, and ABC_3 be an associated system of spherical triangles; $OO_1O_2O_3$ the centres of the inscribed, and $QQ_1Q_2Q_3$ those of the circumscribed, circles. Let QK, Q_1K_1, Q_2K_2, and Q_3K_3 be perpendiculars upon BC; QL, Q_1L_1, Q_2L_2, and Q_3L_3 perpendiculars upon AC, and QM, Q_1M_1, Q_2M_2, and Q_3M_3 perpendiculars upon AB. Then will

$$\begin{aligned}
&\cot AO_2 \cot AO_3 - \cot AO \cot AO_1 = 1\\
&\cot BO_1 \cot BO_3 - \cot BO \cot BO_2 = 1\\
&\cot CO_1 \cot CO_2 - \cot CO \cot CO_3 = 1\\
\text{and,}\quad &\tan Q_2K_2 \tan Q_3K_3 + \tan QK \tan Q_1K_1 = 1\\
&\tan Q_1L_1 \tan Q_3L_3 + \tan QL \tan Q_2L_2 = 1\\
&\tan Q_1M_1 \tan Q_2M_2 + \tan QM \tan Q_3M_3 = 1.
\end{aligned}$$

62. *Mr. James Reid, Prior's Salford.*

If two parabolas be described on the same axis so that the vertex of the second may be the focus of the first, and its parameter half that of the other; then if from the points in which normals of the smaller parabola meet the evolute of the larger, as centres, and the respective normals as radii, circles be described, these will all pass through a certain fixed point.

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63. *From the Putney College Examination Papers, July*, 1845.

The ground plan and picture being taken as planes of projection, there are given two points, and a finite line by means of their projections, and a plane by means of its traces. If light emanate from one of the given points, and the given line enlightened by it cast a shadow on the given plane, it is required :—

(1) To construct the projections of the shadow on the planes of projection.

(2) To construct the perspective of that shadow, the eye being in the second given point.

64. *Pen-and-Ink.*

When Captain Cook arrived at the Antipodes of London, he remarked in his journal that he was "as far from his friends as he could be;" was he so?

65. *Mr. Fenwick.*

Prove that the locus of the centres of the innumerable surfaces of the second order that have eight points common in space, is a surface of the second order, and thence deduce the corresponding locus in plano.

66. *Mr. G. W. Hearn, Professor of Mathematics, R. M. College.*

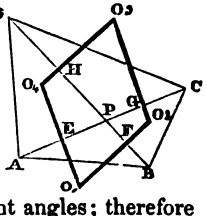
Required the mutual action of two closed voltaic currents of small dimensions on each other.

SOLUTIONS OF MATHEMATICAL EXERCISES.

XLIII. *By* γ.

If about each of the four triangles into which any quadrilateral figure is divided by its diagonals, a circle be described, the centres of these circles are the four angular points of a parallelogram.

[First Solution. *Mr. James Reid, Prior's Salford, Warwickshire.*]



Let ABCD be the quadrilateral, AC and BD its diagonals intersecting each other in P; O_1, O_2, O_3, O_4, the centres of the circles described about the triangles APB, BPC, CPD, DPA. Join O_1O_2, O_2O_3, O_3O_4, O_4O_1.

Because AP is a common chord of the circles whose centres are O_1 and O_4, O_1O_4 cuts AP, that is AC, at right angles. For a similar reason O_2O_3 cuts AC at right angles; therefore O_1O_4 is parallel to O_2O_3. In like manner O_1O_2 is parallel to O_3O_4; wherefore $O_1O_2O_3O_4$ is a parallelogram.

[Second Solution. *Mr. W. H. Levy, Shalbourne.*]

Let ABCD be the quadrilateral, and O_1 O_2 O_3 and O_4 the centres of the circles circumscribed about the triangles APB, BPC, CPD, DPA respectively.

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Draw the lines O_1O_2, O_2O_3, O_3O_4, O_4O_1 cutting the diagonals in F, G, H, E; also conceive the centre of each circle and the angles of its inscribed triangle respectively to be joined.

Then since $O_1A = O_1P$ and $O_4A = O_4P$, the line O_1O_4 cuts AP at right angles in E; and in a similar manner each of the angles at F, G, and H, is found to be a right angle. Now in the quadrilaterals PEO_4H and FO_2GP, take away the sum of two right angles, together with the equal angles at P, and we have angle EO_4H = angle FO_2G, that is, angle $O_1O_4O_3$ = angle $O_1O_2O_3$; and in like manner it may be proved that angle $O_2O_1O_4 = O_2O_3O_4$; consequently $O_1O_2O_3O_4$ is a parallelogram.

[THIRD SOLUTION. *Mr. Thomas Wilkinson, Cheapside, Burnley, Lancashire; and similarly by Messrs. S. Bills, Hawton; and J. W. Elliott, Greatham.*]

Let ABCD be the quadrilateral, AC, BD the diagonals, intersecting in P and dividing the figure into the four triangles, APB, BPC, CPD, DPA. Bisect the lines AP, CP, BP, DP by perpendiculars intersecting each other in O_1, O_2, O_3, and O_4 respectively; these points are evidently the centres of the circles circumscribing the several triangles.

Then since the diagonal AC is divided perpendicularly by O_1O_4, and O_2O_3, O_1O_4 is parallel to O_2O_3. Also, since BD is divided perpendicularly by O_1O_2 and O_3O_4, O_1O_2 is parallel to O_3O_4. Consequently $O_1O_2O_3O_4$ is a parallelogram.

[FOURTH SOLUTION. *Mr. John Laws, Newcastle.*]

Let P be the point of intersection of the diagonals AC, BD of the quadrilateral ABCD; and $O_1O_2O_3O_4$ the centres of the four circles described about the triangles APB, BPC, CPD, DPA respectively. Join O_1O_2 cutting BD in F; also join O_2O_3, O_3O_4, O_4O_1, O_1P, O_1B, O_2P, O_2B. Because $PO_1 = BO_1$ and O_1O_2 common to the triangles PO_1O_2, BO_1O_2, and $O_2P = O_2B$; therefore the angle $PO_1F = BO_1F$; hence the two sides PO_1, O_1F are equal to the two sides BO_1, O_1F, and the included angles are equal; therefore the angle $PFO_1 = BFO_1$, that is O_1O_2 is perpendicular to BD. Similarly O_3O_4 is perpendicular to BD; therefore O_1O_2 is parallel to O_3O_4. In a similar manner it may be proved that O_2O_3 is parallel to O_1O_4; hence $O_1O_2O_3O_4$ is a parallelogram.

Solutions were received from Messrs. G. W. Hearn, Royal Military College, Sandhurst, and Nicholas Smyth, Galway.

XLIV. *James Lockhart, Esq., Thistle Grove, Chelsea.*

If x_1 be one of the roots of the cubic equation $x^3 - bx - c = 0$; then the two other roots can be found from either of the following fractional expressions:—

$$-\frac{\{4{\cdot}5\,c + (b^3 - 6{\cdot}75\,c^2)^{\frac{1}{2}}\}x_1 + b^2}{3bx_1 + 4{\cdot}5c - (b^3 - 6{\cdot}75c^2)^{\frac{1}{2}}}, \text{ or } -\frac{\{4{\cdot}5c - (b^3 - 6{\cdot}75c^2)^{\frac{1}{2}}\}x_1 + b^2}{3bx_1 + 4{\cdot}5c + (b^3 - 6{\cdot}75c^2)^{\frac{1}{2}}}.$$

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[First Solution. *Mr. Thomas Weddle.*]

Let x_1, x_2, and x_3 be the three roots; then we have

$$x_1 + x_2 + x_3 = 0 \quad \ldots\ldots(1).$$
$$x_1 x_2 + x_1 x_3 + x_2 x_3 = -b \quad \ldots\ldots(2).$$
$$x_1 x_2 x_3 = c \quad \ldots\ldots(3).$$

Eliminate x_3 from (2, 3) by means of (1);

$$\therefore x_1^2 + x_1 x_2 + x_2^2 = b \quad \ldots\ldots(4).$$
$$x_1^2 x_2 + x_1 x_2^2 = -c \quad \ldots\ldots(5).$$

Cube (4) and deduct $\frac{27}{4}$ or 6·75 times the square of (5), and extract the square root,

$$\therefore x_1^3 + 1{\cdot}5\, x_1^2 x_2 - 1{\cdot}5\, x_1 x_2^2 - x_2^3 = \pm (b^3 - 6{\cdot}75\, c^2)^{\frac{1}{2}} \ldots\ldots(6).$$

From (5), $4{\cdot}5\, x_1^2 x_2 + 4{\cdot}5\, x_1 x_2^2 = -4{\cdot}5\, c \ldots\ldots(7).$

Subtraction and addition of (6, 7) give,

$$x_1^3 - 3\, x_1^2 x_2 - 6\, x_1 x_2^2 - x_2^3 = 4{\cdot}5\, c \pm (b^3 - 6{\cdot}75\, c^2)^{\frac{1}{2}} \ldots\ldots(8).$$

$$x_1^3 + 6\, x_1^2 x_2 + 3\, x_1 x_2^2 - x_2^3 = -4{\cdot}5\, c \pm (b^3 - 6{\cdot}75\, c^2)^{\frac{1}{2}} \ldots\ldots(9).$$

To (8) multiplied by x_1 add the square of (4),

$$\therefore 2\, x_1^4 - x_1^3 x_2 - 3\, x_1^2 x_2^2 + x_1 x_2^3 + x_2^4 = \{4{\cdot}5\, c \pm (b^3 - 6{\cdot}75\, c^2)^{\frac{1}{2}}\} x_1 + b^2 \ldots\ldots(10).$$

Deduct (9) from (4) multiplied by $3\, x_1$,

$$\therefore 2\, x_1^3 - 3\, x_1^2 x_2 + x_2^3 = 3\, b\, x_1 + 4{\cdot}5\, c \mp (b^3 - 6{\cdot}75\, c^2)^{\frac{1}{2}} \ldots\ldots(11).$$

Divide (10) by (11);

$$\therefore x_1 + x_2 = \frac{\{4{\cdot}5\, c \pm (b^3 - 6{\cdot}75\, c^2)^{\frac{1}{2}}\} x_1 + b^2}{3\, b\, x_1 + 4{\cdot}5\, c \mp (b^3 - 6{\cdot}75\, c^2)^{\frac{1}{2}}}.$$

But by (1), $x_3 = -(x_1 + x_2)$;

$$\therefore x_3 = -\frac{\{4{\cdot}5\, c \pm (b^3 - 6{\cdot}75\, c^2)^{\frac{1}{2}}\} x_1 + b^2}{3\, b\, x_1 + 4{\cdot}5\, c \mp (b^3 - 6{\cdot}75\, c^2)^{\frac{1}{2}}} \ldots\ldots(12).$$

If we had begun by eliminating x_2, we should have found the same expression for the value of x_2, hence the two values of (12) are the two roots x_2 and x_3.

[Second Solution. *Mr. G. W. Hearn, Royal Military College, Sandhurst.*]

Let
$$z_1 = 4{\cdot}5\, c + (b^3 - 6{\cdot}75\, c^2)^{\frac{1}{2}}$$
$$z_2 = 4{\cdot}5\, c - (b^3 - 6{\cdot}75\, c^2)^{\frac{1}{2}}$$

Then $z_1 + z_2 = 9\, c$, $z_1 z_2 = 27\, c^2 - b^3$, and the expressions for the roots may be written

$$-\frac{x_1 z_1 + b^2}{3 b x_1 + z_2}; \quad -\frac{x_1 z_2 + b^2}{3 b x_1 + z_1}.$$

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Taking the sum and product of these we find the former equal to

$$-\frac{27\,bcx_1^2+(27\,c^2+8\,b^3)\,x_1+9\,b^2c}{9\,b^2x_1^2+27\,bc\,x_1+27\,c^2-b^3},$$

and the latter

$$\frac{(27\,c^2-b^3)\,x_1^2+9\,b^2c\,x_1+b^4}{9\,b^2x_1^2+27\,bc\,x_1+27\,c^2-b^3}.$$

If the denominator of these be multiplied by x_1 and we put $x_1^3=b\,x_1+c$, we find that the numerator of the first is reproduced. Also if the denominator of the second be multiplied by x_1^2-b and we put $b\,x_1^2+c\,x_1$ for x_1^4 and $b\,x_1+c$ for x_1^3, we find that the numerator of the second is reproduced.

Hence the two expressions, subject to the condition that x_1 is a root of $x^3-b\,x-c=0$, are equivalent to $-x_1$ and x_1^2-b respectively, and therefore the expressions are the roots of the equation

$$x^2+x_1x+x_1^2-b=0;$$

but this is the quotient resulting from the division of $x^3-b\,x-c$ by $x-x_1$, so that its roots are the other two roots of $x^3-b\,x-c=0$.

XLV. *Mr. Rutherford.*

If in each side of a triangle two points be taken (either all in the sides, or all in the sides produced) at equal distances from the angular points; these six points will be in the same conic section.

[First Solution. *Mr. John Laws; and similarly by Messrs. J. W. Elliott, Greatham; W. H. Levy, Shalbourne; and Samuel Bills, Hawton.*]

Let D, E; F, G; H, I (the diagram for both cases is easily sketched) be the six points taken on the sides AB, BC, CA respectively, of any plane triangle ABC, as specified in the enunciation.

Then $$AD=BE,\ BF=CG,\ CH=AI,$$
$$AE=BD,\ BG=CF,\ CI=AH;$$

therefore $AD.AE.BF.BG.CH.CI=BE.BD.CG.CF.AI.AH$ ..... (1).

Draw the lines ID, EF, GH, and let ID meet BC in L, EF meet AC in M, and GH meet AB in N; then, the sides of the triangle ABC being cut by the transversals ID, EF, GH, we have

$$\left.\begin{aligned}AD.BL.CI&=BD.AI.CL\\AE.BF.CM&=BE.AM.CF\\AN.BG.CH&=AH.BN.CG\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Divide the continued product of (2) by (1) and there results

$$AN.BL.CM=AM.BN.CL\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Therefore the points L, M, N, range in a straight line; and hence (converse of Pascal's theorem) the points D, E, F, G, H, I, lie in the same conic section.

[Second Solution. *Mr. Thomas Weddle.*]

Let ABC be the triangle, $BC=a$, $AC=b$, and $AB=c$.

In the sides AB, AC and BC produced take $AC_1=BC_2=AB_1=CB_2=BA_1=CA_2=m\,a$. Take AC, AB for axes, then a little consideration

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will make it evident that the points C_1, C_2, B_1, B_2, A_1 and A_2 are denoted by $(-ma, o)$, $(c+ma, o)$, $(o, -ma)$, $(o, b+ma)$, $\{-mc, (1+m)b\}$, and $\{(1+m)c, -mb\}$. The conic section passing through the first four points is,

$$(ma+c)(y+ma)(y-ma-b)+(ma+b)(x+ma)(x-ma-c)$$
$$+Bxy+ma(ma+b)(ma+c)=0 \ldots\ldots\ldots\ldots(1).$$

B being an arbitrary constant. Substitute the co-ordinates of the fifth point;

$$\therefore (1+m)bcB=(b-a)(ma+c)\{(1+m)b+ma\}$$
$$+(c-a)(ma+b)\{(1+m)c+ma\}+a(ma+b)(ma+c)\ldots(2).$$

This equation determines B; but if we had substituted the co-ordinates of the sixth point we should have got the very same value of B. Hence if B be eliminated from (1, 2), the resulting equation will represent a conic section which passes through all the six points. The investigation for the case when the points are in the sides is the same after changing the sign of m.

[THIRD SOLUTION. *Mr. James Reid, Prior's Salford; and similarly by Mr. Nicholas Smyth, Galway.*]

Let ABC be a triangle, (x_1, y_1), (x_2, y_2) two points in AB at the distance d_1 from the middle point of AB; (x_3, y_3), (x_4, y_4) points in AC, at the distance d_2 from the middle point of AC; and (x_5, y_5), (x_6, y_6) points in BC at the distance d_3 from the middle point of BC.

Take A as the origin, AB as the axis of x and AC as that of y. Then we shall evidently have,

$$x_1=\frac{c}{2}-d_1,\ y_1=0 \qquad x_4=0,\qquad y_4=\frac{b}{2}+d_2$$
$$x_2=\frac{c}{2}+d_1,\ y_2=0 \qquad x_5=\frac{(\frac{1}{2}a+d_3)c}{a},\ y_5=\frac{(\frac{1}{2}a-d_3)b}{a}$$
$$x_3=0,\qquad y_3=\frac{b}{2}-d_2 \qquad x_6=\frac{(\frac{1}{2}a-d_3)c}{a},\ y_6=\frac{(\frac{1}{2}a+d_3)b}{a}.$$

Assume the equation to a conic section passing through the first five points,

$$Ax^2+By^2+Cxy+Dx+Ey+1=0 \ldots\ldots\ldots\ldots(1).$$

From (1) and the first four of the preceding co-ordinates we obtain

$$A\left(\frac{c}{2}-d_1\right)^2+D\left(\frac{c}{2}-d_1\right)+1=0$$
$$A\left(\frac{c}{2}+d_1\right)^2+D\left(\frac{c}{2}+d_1\right)+1=0$$
$$B\left(\frac{b}{2}-d_2\right)^2+E\left(\frac{b}{2}-d_2\right)+1=0$$
$$B\left(\frac{b}{2}+d_2\right)^2+E\left(\frac{b}{2}+d_2\right)+1=0.$$

From these equations we readily find

$$D=-Ac \text{ and } E=-Bb \ldots\ldots\ldots\ldots(2)$$

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We might determine the values of A, B, D, E in terms of b, c, d_1, d_2, but in the present inquiry it is only necessary to find the ratios of D to A and E to B. These ratios, it is to be observed, are independent of d_1 and d_2.

To determine the remaining coefficient, C, of (1), let us substitute in (1) the values of x_5 and y_5 instead of x and y. The equation will then, by the help of (2), be easily reduced to the form

$$-\frac{(\frac{1}{4}a^2-d_3^2)bc}{a^2}\left\{A\frac{c}{b}+B\frac{b}{c}-C\right\}+1=0,$$

or,
$$-x_5y_5\left(A\frac{c}{b}+B\frac{b}{c}-C\right)+1=0;$$

$$\therefore\quad C=A\frac{c}{b}+B\frac{b}{c}-\frac{1}{x_5y_5}\quad\ldots\ldots\quad\ldots\ldots\ldots(3)$$

Since $x_6y_6=x_5y_5$, it is plain that the value of C will be the same whether the curve be subjected to the condition of passing through the point (x_5, y_5) or (x_6, y_6); therefore, since the other coefficients are independent of the position of these points, the same curve passes through both. It is to be remarked that d_1, d_2, d_3 are independent of each other and may be of any value whatever. When they are less than the halves of the respective sides, the points are in the sides; when greater, in the sides produced.

Cor. 1. If the sides of the triangle are indefinitely diminished it becomes a point; hence if in a plane three straight lines passing through a point are bisected in that point, their extremities are in the same conic section.

Cor. 2. If in a conic section two chords cutting one another are bisected in their point of intersection, every other chord through the same point is bisected; wherefore the only point in which two chords can bisect each other is the centre of the curve.

[FOURTH SOLUTION. *Mr. G. W. Hearn, Royal Military College, Sandhurst.*]

Let CB be axis of x and CA axis of y. Denote the sides of the triangle by a, b, c; CP = CR = BQ = BV = AT = AS = m, where P, Q are in the side a; R, S in b; and T, V in c; and let

$$u=mx+(a-m)(y-m)$$
$$v=my+(b-m)(x-m),$$

so that
$u=0$ is equation to RQ
$v=0$ .. PS.

The equation to a conic section passing through S, P, Q, R will be

$$uv=kxy\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

for this is an equation of the second order and is satisfied by $u=0$, $x=0$; $u=0$, $y=0$; $v=0$, $x=0$; $v=0$, $y=0$; which conditions are necessary and sufficient to insure the curve (1) passing through S, P, Q, R.

The co-ordinates of V are easily found by similar triangles to be

$$x=\frac{a(c-m)}{c},\ y=\frac{bm}{c};$$

and those of T,
$$x=\frac{am}{c},\ y=\frac{b(c-m)}{c}.$$

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The latter system may be deduced from the former by interchanging x and y; a and b. But by such an interchanging the equation (1) remains unaltered, since u and v merely interchange. Hence whether we seek to determine k from the 1st or 2nd systems of values of x and y, we must have the same result; and therefore if the curve (1) be made to pass through V it will also pass through T, and hence P, Q, R, S, T, V are in the same conic section.

Cor. When the points are in the sides produced, m is negative; and it is easily seen that the same mode of reasoning as that above applies when the points are arranged, two and two, in the sides in a more general way so as to be at distances m, m_1, m_2 from the extremities of the sides.

XLVI. *Mr. Rutherford.*

The moment of inertia of a circle radius a, revolving about an axis through its centre, and making an angle α with its plane, is

$$Mk^2 = \frac{\pi a^4}{4}(1 + \sin^2 \alpha)$$

[First Solution. *Mr. G. W. Hearn, R. M. College, Sandhurst.*]

The three moments of inertia of any solid about the three principal axes being A, B, C; if θ, ϕ, Ψ be the angles made by any other axis with the principal axes, the moment of inertia about this is

$$A \cos^2 \theta + B \cos^2 \phi + C \cos^2 \Psi.$$

In the case of the circle we have

$$A = \Sigma mx^2,\ B = \Sigma my^2,\ C = \Sigma mr^2,$$
$$\therefore\ C = \Sigma mr^2 = \Sigma m(x^2 + y^2) = A + B.$$

Also $\Sigma mx^2 = \Sigma my^2$ by symmetry; $\therefore\ A = B = \frac{1}{2}C$.

Also $C = \iint r\, d\theta\, dr . r^2 = \frac{2\pi a^4}{4}$; hence moment required

$$= \frac{\pi a^4}{4}(\cos^2 \theta + \cos^2 \phi + 2 \cos^2 \Psi);$$

but the principal axes being mutually at right angles, we have the condition

$$\cos^2 \theta + \cos^2 \phi + \cos^2 \Psi = 1.$$

Also, $$\alpha = \frac{\pi}{2} - \Psi \quad \therefore\ \cos \Psi = \sin \alpha;$$

$$\therefore \text{ moment} = \frac{\pi a^4}{4}(1 + \sin^2 \alpha)$$

[Second Solution. *Mr. J. W. Elliott: and similarly by Mr. Weddle.*]

Let Ox, Oy, Oz be rectangular axes, originating at O the centre of the circle, and OA the axis, in reference to which the moment of inertia is to be determined. Then *Poisson's Mécanique*, Art 375,

$$Mk^2 = A \cos^2 \alpha' + B \cos^2 \beta' + C \cos^2 \gamma' \ldots\ldots\ldots\ldots (1)$$

where A, B, C denote the moments of inertia with respect to the axes of x, y, z, and α', β', γ', the angles which OA makes with these axes. By the usual methods, we find

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$$A = \frac{\pi a^4}{4},\ B = \frac{\pi a^4}{4},\ C = \frac{\pi a^4}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

wherefore by substitution in (1), and from the well known equation

$$\cos^2 a' + \cos^2 \beta' + \cos^2 \gamma' = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

we have

$$Mk^2 = \frac{\pi a^4}{4}(1 + \cos^2 \gamma') = \frac{\pi a^4}{4}(1 + \sin^2 a) \ldots\ldots\ldots (4)$$

the moment with respect to the axis OA, which was required.

When OA is equally inclined to the *principal* axes, that is, when $a' = \beta' = \gamma'$, equation (1) becomes

$$Mk^2 = (A + B + C)\cos^2 a';$$

but from (3),

$$3\cos^2 a' = 1, \text{ or } \cos^2 a' = \tfrac{1}{3};$$

$$\therefore\ Mk^2 = \tfrac{1}{3}(A + B + C) = \tfrac{1}{3}\pi a^4 \ldots\ldots\ldots\ldots\ldots (5)$$

From (2) there obtains $A + B = C$; and comparing with (5), we find

$$Mk^2 = \tfrac{2}{3} C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

which holds, not only for the circle, but also for any other plane figure, as may be briefly shown. For let x, y be the co-ordinates of an element dm of the plane; then

$$C = \Sigma\, dm\,(x^2 + y^2) = \Sigma\, dm.\, x^2 + \Sigma\, dm.\, y^2 = A + B;$$

consequently $\qquad Mk^2 = \tfrac{1}{3}(A + B + C) = \tfrac{2}{3} C,$

which may be thus expressed:

The moment of inertia of any plane figure, revolving about an axis having the same origin as the principal axes and equally inclined to them, is equal to two-thirds of the *greatest* of the moments about those principal axes.

From (4), when $a = 0$, or $a = \frac{1}{2}\pi$, we have the moments as expressed in (2). There are also other properties, such as when $a' = \beta'$, or $\beta' = \gamma'$, or $\gamma' = a'$, which give neat expressions for Mk^2.

[THIRD SOLUTION. *Mr. James Reid.*]

The moment of inertia of a heavy particle is a function of its distance from the axis; therefore the moment of inertia of a particle is not changed when the particle is transferred from one point to another in a direction parallel to the axis. Let the given circle be projected on a plane perpendicular to the axis, every particle in the circle being transferred to this plane in a direction parallel to the axis. The projection will be an ellipse of which the axis major is $2a$, and the axis minor $2a \sin a$. The moment of inertia of the ellipse is the same as that of the circle, and if the density of the circle $= 1$, that of the ellipse $= \dfrac{1}{\sin a}$. Now a and b being the semi-axes of an ellipse, its moment of inertia about a straight line perpendicular to its plane is

$$= M.\ \frac{a^2 + b^2}{4} \quad \text{(Earnshaw's Dynamics, Art. 183.)}$$

In the case we are considering, $M = \pi a^2$, and $b = a \sin a$;

$$\therefore \qquad Mk^2 = \frac{\pi a^4}{4}(1 + \sin^2 a).$$

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[FOURTH SOLUTION. *Mr. Weddle; and similarly by Mr. Bills.*]

Take the centre O for the origin of rectangular co-ordinates, the axis of z being perpendicular to the plane of the circle, and that of x the projection of the given line. Let P or (x, y) be the position of any element $dxdy$ of the circle; draw $PQ = y$, perpendicular to the axis of x, and PR perpendicular to the given line, and join QR. The angles PRO, PQR and ORQ are evidently right angles; hence $QR = x \sin a$, and $PR^2 = PQ^2 + QR^2 = y^2 + x^2 \sin^2 a$;

$\therefore$ moment of inertia $= Mk^2 = \iint PR^2 . dm = \iint y^2\, dxdy + \sin^2 a \iint x^2\, dx\, dy$.

Integrate first with respect to y between the limits $y = -\sqrt{(a^2 - x^2)}$, and $y = \sqrt{(a^2 - x^2)}$, and then with respect to x between the limits $x = -a$, and $x = a$; we finally get,

$$Mk^2 = \frac{\pi a^4}{4}(1 + \sin^2 a).$$

Cor. 1. The moment of inertia of a circle revolving about a diameter is (since $a = 0$)

$$Mk^2 = \frac{\pi a^4}{4}.$$

Cor. 2. The moment of inertia of a circle revolving round an axis through its centre perpendicular to its plane is (since $a = 90°$)

$$Mk^2 = \frac{\pi a^4}{2}.$$

Good solutions were received from Messrs. R. H. Wright, London, and Nicholas Smyth, Galway.

XLVII. *By ϕ.*

Over a parabola whose axis is vertical a string passes, at the extremities of which two weights w and w' are suspended. Shew that if y and y' be the co-ordinates of the points where they rest in equilibrium, we shall have

$$\left(\frac{w'}{y}\right)^2 - \left(\frac{w}{y'}\right)^2 = \frac{w^2 - w'^2}{4m^2}.$$

[FIRST SOLUTION. *Mr. Weddle; and similarly by Mr. James Reid.*]

If a and a' denote the angles which the axes make with the tangents at the points where the weights rest, we shall have,

$$\cos a = \frac{y}{\sqrt{y^2 + 4m^2}}, \text{ and } \cos a' = \frac{y'}{\sqrt{y'^2 + 4m^2}}.$$

Now, that there may be equilibrium, the tangential components of the weights must be equal;

$$\therefore\ w \cos a = w' \cos a',$$

or,
$$\frac{yw}{\sqrt{y^2 + 4m^2}} = \frac{y'w'}{\sqrt{y'^2 + 4m^2}};$$

that is,
$$\frac{w'}{y} \cdot \sqrt{y^2 + 4m^2} = \frac{w}{y'} \sqrt{y'^2 + 4m^2};$$

$$\therefore \text{ squaring, } w'^2 + 4m^2 \left(\frac{w'}{y}\right)^2 = w^2 + 4m^2 \left(\frac{w}{y'}\right)^2;$$

$$\therefore \left(\frac{w'}{y}\right)^2 - \left(\frac{w}{y'}\right)^2 = \frac{w^2 - w'^2}{4m^2}.$$

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[SECOND SOLUTION. *By H. R. L.; and similarly by Mr. N. Smyth.*]

Let T be the tension of the string; x, s the corresponding abscissa and arc to the ordinate y; x', s' to y'; and P and P' the points where w and w' are at rest. Then for the equilibrium of the points P and P', resolving forces along the tangents at these points, we have

$$T = w\frac{dx}{ds} \ldots\ldots(1), \quad T = w'\frac{dx'}{ds'} \ldots\ldots(2).$$

$$\text{But } y^2 = 4mx, \quad \therefore \frac{dy}{dx} = \frac{2m}{y}, \frac{ds}{dx} = \sqrt{\left(1 + \frac{4m^2}{y^2}\right)}.$$

$$\text{Similarly,} \quad \frac{ds'}{dx'} = \sqrt{\left(1 + \frac{4m^2}{y'^2}\right)}.$$

Hence equating (1) and (2), and substituting these values,

$$\frac{w\,y}{\sqrt{(y^2+4m^2)}} = \frac{w'\,y'}{\sqrt{(y'^2+4m^2)}}, \text{ or } w^2y^2(y'^2+4m^2) = w'^2y'^2(y^2+4m^2);$$

$$\therefore \left(\frac{w'}{y}\right)^2 - \left(\frac{w}{y'}\right)^2 = \frac{w^2 - w'^2}{4m^2}.$$

The problem may also be solved by virtual velocities.

[THIRD SOLUTION. *Mr. Samuel Bills, Hawton.*]

Let WVW' be a parabola, V the vertex, and VNN' the vertical axis, NW, N'W' being the ordinates at the points where the weights w and w' are in equilibrium.

Draw the tangents WP, W'P', meeting the axis in P and P'; and intersecting each other in I. Now it is evident that the two weights w and w' may be regarded as resting on the inclined planes IW, IW', and that the conditions of equilibrium will be the same as if they were suspended by a string passing round WIW'. Hence by the principles of Mechanics, the equation of equilibrium is

$$w \sin \text{IWN} = w' \sin \text{IW'N'} \ldots\ldots\ldots\ldots(a)$$

Again, $y^2 = 4mx$, being the equation of the parabola, we readily find by the properties of the figure, that

$$\sin^2 \text{IWN} = \sin^2 \text{PWN} = \frac{y^2}{y^2+4m^2}, \text{ and } \sin^2 \text{IW'N'} = \frac{y'^2}{y'^2+4m^2}.$$

The equation (a) therefore becomes

$$\frac{w^2y^2}{y^2+4m^2} = \frac{w'^2y'^2}{y'^2+4m^2}, \text{ or } \left(\frac{w'}{y}\right)^2 - \left(\frac{w}{y'}\right)^2 = \frac{w^2-w'^2}{4m^2};$$

which is the beautiful relation given in the question.

[FOURTH SOLUTION. *Mr. R. H. Wright, London.*]

The equation of equilibrium is

$$w - w'\frac{dy}{dx} = 0.$$

$$\text{Let } y^2 = 4mx, \quad \therefore \quad y\frac{dy}{dx} = 2m, \text{ or } \frac{dy}{dx} = \frac{2m}{y}.$$

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Similarly, $\frac{dy'}{dx'} = \frac{2m}{y'}$; hence we have

$$w - w' \frac{2m}{y} = 0, \text{ or } \frac{w^3}{4m^2} = \left(\frac{w'}{y}\right)^2,$$

$$w' - w \frac{2m}{y'} = 0, \text{ or } \frac{w'^2}{4m^2} = \left(\frac{w}{y'}\right)^2;$$

$$\therefore \left(\frac{w'}{y}\right)^2 - \left(\frac{w}{y'}\right)^2 = \frac{w^3 - w'^3}{4m^2}.$$

The solutions of Messrs. Hearn and Elliott were omitted with much regret.

XLVIII. *By Pen-and-Ink, Charlton.*

Let any two planes X, Y intersect in a line, and three points P, Q, R in that line be taken: draw any three lines PP_1, QQ_1, RR_1 in the plane X, and any three PP_2, QQ_2, RR_2 in the plane Y: let PP_1, QQ_1, RR_1 mutually intersect in a_1, b_1, c_1, and PP_2, QQ_2, RR_2, corresponding to the former ones, intersect in a_2, b_2, c_2: then the lines $a_1 a_2$, $b_1 b_2$, $c_1 c_2$ will pass through the same point, but they can never be all in the same plane, except when the planes X and Y coincide.

[First Solution. *Mr. Weddle; and similarly by Mr. G. W. Hearn.*]

Since $a_1 a_2$ is the intersection of the planes P_1PP_2 and Q_1QQ_2; b_1b_2, the intersection of R_1RR_2 and P_1PP_2; and $c_1 c_2$ the intersection of Q_1QQ_2 and R_1RR_2; it follows that $a_1 a_2$, $b_1 b_2$, $c_1 c_2$, are the intersections of three planes, and therefore either pass through the same point or are parallel. Also $a_1 a_2$, $b_1 b_2$, $c_1 c_2$ cannot be in the same plane unless the planes P_1PP_2, Q_1QQ_2, R_1RR_2 coincide, which can only be the case when X and Y coincide.

[Second Solution. *Messrs. John Laws, and Nicholas Smyth.*]

(Whose solutions are almost identical.)

Let P be the origin of oblique co-ordinates; PP_1, PP_2, PR, the axes of x, y, z, respectively; also draw the lines Ra_1, Ra_2, Qb_1, Qb_2.

Put $Pa_1 = a$, $Pa_2 = \beta$, $PQ = \gamma$, $Pb_1 = a_1$, $Pb_2 = \beta_1$, $PR = \gamma_1$.

Then, the equations of the planes a_1Qb_2, b_1Ra_2, a_2Qb_1, b_2Ra_1, are as follows:

$$(a_1Qb_2) \ldots \frac{x}{a} + \frac{y}{\beta_1} + \frac{z}{\gamma} = 1 \ldots (1) \qquad (b_1Ra_2) \ldots \frac{x}{a} + \frac{y}{\beta} + \frac{z}{\gamma_1} = 1 \ldots (2)$$

$$(a_2Qb_1) \ldots \frac{x}{a_1} + \frac{y}{\beta} + \frac{z}{\gamma} = 1 \ldots (3) \qquad (b_2Ra_1) \ldots \frac{x}{a} + \frac{y}{\beta_1} + \frac{z}{\gamma_1} = 1 \ldots (4)$$

Now c_1, c_2, are points in the planes (1) (2), (3) (4), respectively; let them be denoted by $(x', 0, z')$, $(0, y'', z'')$, and we then obtain the following equations for the determination of the co-ordinates of c_1, c_2; viz.

$$\frac{x'}{a} + \frac{z'}{\gamma} = 1, \quad \frac{x'}{a_1} + \frac{z'}{\gamma_1} = 1 \ldots \ldots \ldots \ldots \ldots (5)$$

$$\frac{y''}{\beta} + \frac{z''}{\gamma} = 1, \quad \frac{y''}{\beta_1} + \frac{z''}{\gamma_1} = 1 \ldots \ldots \ldots \ldots \ldots (6)$$

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The co-ordinates of c_1, c_2, are therefore

$$\left.\begin{aligned} x' &= \frac{aa_1(\gamma_1 - \gamma)}{a\gamma_1 - a_1\gamma}, \qquad z' = \frac{\gamma\gamma_1(a - a_1)}{a\gamma_1 - a_1\gamma} \\ y' &= \frac{\beta\beta_1(\gamma_1 - \gamma)}{\beta\gamma_1 - \beta_1\gamma}, \qquad z'' = \frac{\gamma\gamma_1(\beta - \beta_1)}{\beta\gamma_1 - \beta_1\gamma} \end{aligned}\right\} \ldots\ldots\ldots (7)$$

Also, the equations of the line $c_1\, c_2$ are

$$x - x' = \frac{x'}{z' - z''}(z - z'), \text{ and } y = \frac{-y''}{z' - z''}(z - z') \ldots\ldots (8)$$

and by making $z = 0$ in (8) we obtain the co-ordinates of the point where $c_1\, c_2$ pierces the plane of xy, viz.

$$x = \frac{-x'z''}{z' - z''}, y = \frac{z'y''}{z' - z''} \ldots\ldots\ldots\ldots\ldots (9)$$

and which, by substituting for x', z', y'', z'', their values as given in (7), become

$$x = -\frac{aa_1(\beta - \beta_1)}{a\beta - a_1\beta}, y = \frac{\beta\beta_1(a - a_1)}{a\beta_1 - a_1\beta} \ldots\ldots\ldots (10)$$

Again, the equations of the lines $a_1\, a_2$, $b_1\, b_2$, are

$$(a_1\, a_2)\ldots\ldots \frac{x}{a} + \frac{y}{\beta} = 1, (b_1\, b_2)\ldots\ldots \frac{x}{a_1} + \frac{y}{\beta_1} = 1,$$

from which we find the co-ordinates of their point of intersection to be

$$x = -\frac{aa_1(\beta - \beta_1)}{a\beta_1 - a_1\beta}, y = \frac{\beta\beta_1(a - a_1)}{a\beta_1 - a_1\beta} \ldots\ldots\ldots\ldots (11)$$

Equations (10 and (11) being identical, the lines $a_1\, a_2$, $b_1\, b_2$, $c_1\, c_2$, therefore, intersect in one point.

Scholium, by the Proposer.

From the contemplation of the properties of figures situate in space we may often deduce with great simplicity, properties of plane figures which it would require considerable labour to demonstrate in reference to them initially viewed as plane figures. The theorem of Monge relative to the mutual tangents to three circles is a case in point; and Chasles has enumerated a few others which flow from a glance at the epures of descriptive geometry. In the Gentleman's Diary, for 1832 is also another case of a theorem of some difficulty to establish by Geometry, which is at once apparent from the drawing of a problem in the Geometry of Monge; and evidently so deduced originally by the proposer: and the theorem here given will furnish several properties, mutually converse of each other, in a precisely similar manner. We have only to take advantage of the indeterminate angle formed by the planes X and Y of this theorem, and conceive the planes to become identical; when we shall have the following as one of the theorems referred to :—

THEOREM. *If two triangles $a_1b_1c_1$, $a_2b_2c_2$ lying in one plane, have their corresponding sides a_1b_1 and a_2b_2 intersecting in P, b_1c_1 and b_2c_2 intersecting in R, and c_1a_1, c_2a_2 in Q, so that P,Q,R are in one line: then a_1a_2, b_1b_2, c_1c_2 will meet in one point D.*

The young geometer will do well to exercise himself in the demonstration of this theorem by any independent method that may occur to him: perhaps the *transversal method* will furnish the simplest method that can be devised.

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L. *By* γ.

If in a hexagon described about a conic section, the first, third, and fifth sides be produced till they form a triangle, and in like manner the second, fourth, and sixth sides till they form another triangle; then the lines joining the opposite pairs of the angular points of these triangles intersect upon the three diagonals of the hexagon.

[First Solution. *Mr. G. W. Hearn.*]

This is an immediate consequence of the formulæ (a) in my solution to exercise 37, p. 326, vol. i. Mathematician.

Let us then resume the formulæ,

$$\left.\begin{aligned} u_1 - v_2 &= h\,(v_1 - u_3) \\ u_2 - v_3 &= k\,(v_2 - u_3) \\ u_3 - v_1 &= l\,(v_3 - u_1) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

By sketching the figure it will be seen that $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, are the equations to the first, third, and fifth sides; and $v_1 = 0$, $v_2 = 0$, $v_3 = 0$, those of the sides respectively opposite to them.

Now $u_2 + ku_3 = 0$ is the equation to a line through the intersection of u_2 and u_3. But by the preceding conditions, this equation is identical with $v_3 + kv_2 = 0$, a straight line through v_2 and v_3. Hence, either of those equations represents the line joining the intersection of u_2, u_3 with that of v_2, v_3. Similarly, $u_3 + lu_1 = 0$, or $v_1 + lv_3 = 0$, represents the line joining the intersection of u_1, u_3 with that of v_1, v_3.

Now combining $u_3 + lu_1 = 0$, and $v_3 + kv_2 = 0$,

$$ku_3 - lv_3 + lk\,(u_1 - v_2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\beta)$$

And combining $v_1 + lv_3 = 0$, and $u_2 + ku_3 = 0$,

$$ku_3 - lv_3 - (v_1 - u_2) = 0.$$

Subtracting this from (β) we have

$$v_1 - u_2 + lk\,(u_1 - v_2) = 0.$$

But by the first condition, $u_1 - v_2 = h\,(v_1 - u_2)$.

Hence $$(1 + lhk)(v_1 - u_2) = 0,$$

or $$v_1 - u_2 = 0, \text{ or } u_1 - v_2 = 0;$$

either of which represents the diagonal of the hexagon joining v_1u_2, u_1v_2, and consequently the lines joining the intersections of

$$u_2u_3;\ v_2v_3$$
$$u_1u_3;\ v_1v_3$$
$$v_1u_2;\ v_2u_1$$

intersect in the same point.

The same may be proved of the other triads of lines.

Cor. Since the equation (β) also leads to the equation $ku_3 - lv_3 = 0$, it follows that this represents a straight line through r (p, q, r being the points of intersection on the diagonals) and through the intersection of u_3, v_3, two opposite sides of the hexagon. The other combinations give $hu_2 - kv_2 = 0$, and $lu_1 - hv_1 = 0$; the first representing a line through q and u_2, v_2; the other a line through p and u_1, v_1. By combining the conditions of identity with these equations we find that any one of them is a consequence of the other two, and hence the three lines represented by them intersect in the same point.

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The conditions of identity (α) form the readiest and most convenient means of proving any properties that may be predicated of this curious figure. They are the analytical expressions of the condition that the three diagonals of the hexagon intersect in the same point.

[SECOND SOLUTION. *Mr. Thomas Weddle.*]

Let ABCDEF be a hexagon whose sides touch a conic section; let the first, third, and fifth sides (AB, CD, EF) when produced from the triangle MOQ, and the second, fourth, and sixth (BC, DE, and FA) the triangle LNP. Since the sides of the hexagon MCNPFQ touch the conic section, its diagonals MP, QN, and CF intersect in the same point (by Brianchon's theorem); that is, the lines MP and QN intersect upon CF one of the diagonals of the hexagon ABCDEF. Also since the sides of the hexagons LBMOEP and LAQODN touch the conic section, it follows, that, LO and MP intersect upon BE, and LO and QN upon AD.

[THIRD SOLUTION. *Mr. John Laws.*]

Let A, B, C, D, E, F be the six angular points of the circumscribed hexagon, and L, M, N, O, P, Q the six angles of the two triangles formed by producing the alternate sides of the hexagon. Draw the lines LO, MP, NQ, and let LO, MP intersect in R, MP, NQ, in S, and NQ, LO, in T. Now (Mathematician, vol i. p. 326), the six points L, M, N, O, P, Q, lie in a conic section; also in this conic section we have the inscribed hexagons, PLOMQN, NLOQMP, OMPLNQ, whose opposite sides intersect in the three triads of points A, T, D; B, R, E; C, S, F. These three triads of points therefore (Pascal's theorem) range respectively in straight lines, and consequently the points T, R, S are on the diagonals AD, BE, CF of the hexagon ABCDEF.

Correct solutions were also received from Messrs. S. Bills, Hawton; Nicholas Smyth, Galway; and R. H. Wright, London.

LI. *Mr. Fenwick.*

If any one of the first six coefficients of the equation

$$az^2+by^2+cx^2+2dxy+2exz+2fyz+2gz+2hy+2kx+1=0,$$

be arbitrary and independent of the eight remaining coefficients, these eight coefficients will be functions of the co-ordinates of the eight points through which all the surfaces thus represented pass.

[FIRST SOLUTION. *By the Proposer.*]

Let

$$u=ax^2+by^2+cx^2+2dxy+2exz+2fyz+2gz+2hy+2kx+1=0..(1),$$
$$u'=a'x^2+b'y^2+c'x^2+2d'xy+2e'xz+2f'yz+2g'z+2h'y+2k'x+1=0..(2),$$

be the equations of any two surfaces of the second degree which have eight points common in space; then *every* third surface of the second degree passing through the same eight points has for its equation

$$u+\lambda u'=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

λ being an arbitrary multiplier.

For, according as λ varies, so not merely *one* surface but *every one* of

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the indefinite surfaces of the second degree through the same eight points may be expressed by (3).

Again, if we modify (3) by the relations

$$b = b',\ c = c',\ d = d',\ e = e',\ f = f',\ g = g',\ h = h',\ k = k' \ldots . (4),$$

that equation will still represent an infinite number of surfaces of the second degree that have eight points common; the eight points now, however, will have some *particular* position.

Hence substituting in (3) the relations (4), dividing by $(1 + \lambda)$, and writing a_1 for $\frac{a + \lambda a'}{1 + \lambda}$, we get

$$a_1 z^2 + by^2 + cx^2 + 2dxy + 2exz + 2fyz + 2gz + 2hy + 2kx + 1 = 0 \ldots . (5)$$

This is obviously the equation of a surface of the second degree in which the *coefficient of z^2 is arbitrary.*

By altering the relations (4), we should get, successively, equations of surfaces in which the coefficients b, c, *etc.*, are arbitrary. Hence

If any one of the coefficients of the equation

$$az^2 + by^2 + cx^2 + 2dxy + 2exz + 2fyz + 2gz + 2hy + 2kx + 1 = 0,$$

be arbitrary and independent of the remaining coefficients, this equation will then represent an infinite number of surfaces that have eight points common, having in each case some particular position.

Let us next assign the position of the eight points that are common to the surfaces thus represented.

Now for *every* value of the coefficient of z^2 in (5), there exists the following curve of the second degree in the plane of xy,

$$by^2 + cx^2 + 2dxy + 2hy + 2kx + 1 = 0.$$

Let $x_1y_1,\ x_2y_2,\ x_3y_3,\ x_4y_4,\ x_5y_5,$

be the co-ordinates of five points in this curve, and we have

$$\begin{aligned}
by_1^2 + cx_1^2 + 2dx_1y_1 + 2hy_1 + 2kx_1 + 1 &= 0,\\
by_2^2 + cx_2^2 + 2dx_2y_2 + 2hy_2 + 2kx_2 + 1 &= 0,\\
by_3^2 + cx_3^2 + 2dx_3y_3 + 2hy_3 + 2kx_3 + 1 &= 0,\\
by_4^2 + cx_4^2 + 2dx_4y_4 + 2hy_4 + 2kx_4 + 1 &= 0,\\
by_5^2 + cx_5^2 + 2dx_5y_5 + 2hy_5 + 2kx_5 + 1 &= 0.
\end{aligned}$$

By these we can express the coefficients b, c, d, h, k, in terms of the co-ordinates of the five points.

Again, the equations of tangent planes at the points specified above, are (denoting the vertex of the enveloping cone by $\alpha\ \beta\ \gamma$, through which all these tangent planes evidently pass),

$$\begin{aligned}
(fy_1 + ex_1 + g)\gamma + (by_1 + dx_1 + h)\beta + (cx_1 + dy_1 + k)\alpha + hy_1 + kx_1 + 1 &= 0,\\
(fy_2 + ex_2 + g)\gamma + (by_2 + dx_2 + h)\beta + (cx_2 + dy_2 + k)\alpha + hy_2 + kx_2 + 1 &= 0,\\
(fy_3 + ex_3 + g)\gamma + (by_3 + dx_3 + h)\beta + (cx_3 + dy_3 + k)\alpha + hy_3 + kx_3 + 1 &= 0,\\
(fy_4 + ex_4 + g)\gamma + (by_4 + dx_4 + h)\beta + (cx_4 + dy_4 + k)\alpha + hy_4 + kx_4 + 1 &= 0,\\
(fy_5 + ex_5 + g)\gamma + (by_5 + dx_5 + h)\beta + (cx_5 + dy_5 + k)\alpha + hy_5 + kx_5 + 1 &= 0.
\end{aligned}$$

As these all pass through the same point, and as three planes are sufficient to determine a point, these equations are equivalent to *three independent equations* which will express e, f, g, in terms of the co-ordinates of the five points, and α, β, γ.

But since $\alpha\ \beta\ \gamma$ is the pole of the plane of contact (the plane of xy in this

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case), in which are the points x_1y_1, *etc.*, the polar therefore of the point $\alpha\beta\gamma$ gives the equations

$$(b\beta + f\gamma + da + h)y_1 + (ca + e\gamma + d\beta + k)x_1 + g\gamma + h\beta + ka + 1 = 0,$$
$$(b\beta + f\gamma + da + h)y_2 + (ca + e\gamma + d\beta + k)x_2 + g\gamma + h\beta + ka + 1 = 0,$$
$$(b\beta + f\gamma + da + h)y_3 + (ca + e\gamma + d\beta + k)x_3 + g\gamma + h\beta + ka + 1 = 0,$$

which shew that α, β, γ are functions also of the co-ordinates. It follows, then, that all the coefficients of (5) after that of z^2 are functions of the co-ordinates of the points that are common to all the surfaces. The eight points, in this case, are evidently in the same plane, the plane of xy; three of the five points which determine a curve of the second degree in that plane being *double* points, that is, points of contact as well as points of intersection.

A similar mode of investigation is applicable when b, or c is the arbitrary. In the former case, the surfaces have a common *curve of contact* in the plane of xz, in the latter, the curve of contact is in the plane of yz. Next let d be arbitrary in the equation

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz + 2gz + 2hy + 2kz + 1 = 0.$$

Then for all values of d we have

$$ax^2 + by^2 + 2fyz + 2gz + 2hy + 1 = 0,$$

and

$$ax^2 + cz^2 + 2exz + 2gz + 2kx + 1 = 0.$$

The eight points are therefore in two planes, two being in the line of intersection of the planes.

It would be easy to shew also, as in the former part, that the other coefficients are functions of the points common to all the surfaces. And similarly for the coefficients e and f.

The same properties, it would seem, do not belong to the coefficients g, h, k.

It appears from the preceding discussion, that if in any inquiry having reference to surfaces that have eight points common, we can shew by the method of *arbitrary coefficients*, that a property holds in each case, that is, when each coefficient is arbitrary, all we prove is that it is true for some particular positions of the eight points. It is indeed very probable, that a property thus investigated, is *generally* true; but a safer mode of demonstration in such cases is by an arbitrary multiplier (see p. 12). It will be obvious that if a theorem fail by this method for any one coefficient, it must fail in general.

We have in this investigation, another instance of the want of analogy between a property in plano and its corresponding one in space (see the paper at p. 12, just referred to, and vol. i. p. 334).

For let $u = ay^2 + 2bxy + cx^2 + 2dy + 2ex + 1 = 0$ .....(6),

$u' = a'y^2 + 2b'xy + c'x^2 + 2d'y + 2e'x + 1 = 0$.....(7),

be the equations of two curves of the second degree having four points common in a plane. Then

$$u + \lambda u' = 0 \text{ .....(8)},$$

is any third curve passing through the same four points. If now we affect (8) with the relations, $a = a'$, $c = c'$, $d = d'$, $e = e'$, that equation, which becomes

$$ay^2 + \frac{2(b + \lambda b')}{1 + \lambda}xy + cx^2 + 2dy + 2ex + 1 = 0,$$

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will still represent an infinite number of curves that pass through *any* four points in a plane; inasmuch as the co-ordinate axes (in which are now the four points) can be made to pass through any four points in a plane. It follows from this, that if the coefficient of xy of the equation

$$ay^2 + 2bxy + cx^2 + 2dy + 2ex + 1 = 0,$$

be *arbitrary*, this equation will represent an infinite number of curves of the second degree which pass through *any* four points in a plane.

At page 241, vol. 1, of the Mathematician, in a discussion by arbitrary coefficients, I have committed an oversight, which, it is desirable, I should now notice.* I have said that if d be indeterminate in the equation

$$az^2 + by^2 + cx^2 + 2dxy + 2exz + 2fyz + 2gz + 2hy + 2kx + 1 = 0,$$

then the diameters in respect of this surface conjugate to sections made by planes parallel to a given plane, meet in the axis of z. This is not the case; for when d is indeterminate, the equations of a diameter are

$$(f + ma)\,z + (h + mg) = 0,$$
$$(e + na)\,z + (k + ng) = 0.$$

Now, as we do not know that the values of z are equal in these equations, we cannot infer that the diameters intersect in the same point. Similar remarks apply to the coefficients e and f.

I take this opportunity to notice the objections of a correspondent in respect of my demonstration of Pascal's hexagramme (p. 132, vol. I.), by poles and polars. The same result, he observes, would be got by taking any six points in a plane. It is true that the same result might be obtained, but we should not be justified in equating to zero each factor of that result as I have done. This we could do only when each factor is indeterminate in the ordinary sense, that is, when each factor involves variables that are functions of each other.

Referring to the article in question, it will be seen, that by combining the equations (28) and (29) we get the equation (31) of a line through their intersection. Again, by combining (28) and (30) we get the equation (32) of a line through the intersection of (28, 30). Lastly (to connect the system completely) we combine (31) and (32) and we have

$$(2ay + bx + d)\left(\frac{y_1}{x_1} - \frac{y_2}{x_2}\right) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (R),$$

which is the equation of a line passing through the intersection of (31, 32). Now as x and y are the current co-ordinates of the line (R), the first factor is indeterminate. The second is also an indeterminate factor of the kind specified. For the demonstration is quite general by supposing five (A, B, C, E, F), of the six points of the inscribed hexagon given, and the sixth (D), variable in the curve, which we shall denote by X Y. Hence the point K $(x_1\ y_1)$ being determined by the lines AF, CD, will involve X, Y only as variable quantities. Similarly, the co-ordinates $x_2\ y_2$ will involve

* In the investigation just referred to, it is shewn that the diameters in question pass through the same point, in some cases, and in others, through several indeterminate points. This failure, as to the general case, would have been stated more explicitly, but for the oversight here adverted to. I would observe in connection with this subject, that the method by arbitrary coefficients, applies with great facility in that discussion. We prove, very readily, the failure of the property in some particular cases, and hence its failure in general.

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X, Y only as variables. But as D is a point in the curve, X and Y are implicit functions of each other, and therefore

$$2ay + bx + d = 0, \text{ and } \frac{y_1}{x_1} - \frac{y_2}{x_2} = 0;$$

a conclusion that we should not arrive at, if D were *any arbitrary point* in the plane.

[SECOND SOLUTION. *Mr. G. W. Hearn.*]

We will commence with one of the coefficients d, e, f, as d. The difference of two equations differing only in respect of this coefficient will be $2(d-d')xy = 0$, or $xy = 0$, and therefore the points of common intersection make $x = 0$, or $y = 0$; so that these points are necessarily in the planes zy and zx. The points then being taken with due generality there may be four in each plane, or five in one and three in the other.

Making alternately $x = 0$, $y = 0$, in the given equation, the two equations

$$az^2 + by^2 + 2fyz + 2gz + 2hy + 1 = 0,$$
$$az^2 + cx^2 + 2exz + 2gz + 2kx + 1 = 0,$$

will furnish each of them four equations from which the eight coefficients may be determined in terms of the co-ordinates of the points common to the surfaces, or else one equation will furnish five and the other three equations which will in like manner determine the coefficients.

Next, let a one of the coefficients a, b, c, *etc.*, be arbitrary. The difference of two equations will be

$$(a - a')z^2 = 0, \text{ or } z = 0;$$

so that all the points of common intersection must be in the plane xy.

Putting $z = 0$, we have

$$by^2 + cx^2 + 2dxy + 2hy + 2kx + 1 = 0 \ldots\ldots\ldots\ldots (1),$$

and therefore all the surfaces pass through this conic section, since it does not involve the arbitrary coefficient a.

If now we take the equation to a tangent plane at a point $x'y'z'$ and make $z' = 0$, the term in a being azz', will disappear. Hence the several surfaces not only intersect the plane xy in the curve (1), but also have the same enveloping cone, of which (1) is the curve of contact.

Now if the curve (1) pass through four given points, and if the tangent planes to the surfaces at those points contain given straight lines, the remaining eight coefficients $b\ c\ d\ e\ f\ g\ h\ k$ may be determined in terms of the four systems of co-ordinates of the four points, and the direction angles of the four straight lines. But, if we conceive four points infinitely near respectively to the four former points in the curve (1), the cosines of the direction angles may be expressed as functions of the co-ordinates of the two systems of four points, and therefore in this sense, $b\ c\ d\ e\ f\ g\ h\ k$ may be found as functions of the co-ordinates of eight points common to all the surfaces. If five points be taken in (1), it will be sufficient that the tangent planes to the surfaces at three of these points contain given lines. In this case the coefficients $b\ c\ d\ h\ k$ will be completely determined by the co-ordinates of the five points of (1), and e, f, g will be found in terms of $b\ c\ d\ h\ k$ and the direction cosines of the three straight lines, that is, in terms

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of the co-ordinates of the five points and those of three points infinitely near to three of them.

If one of the three g, h, k be arbitrary as g, we have as before $z = 0$, for the points common, and the surfaces all intersect in the curve (1), but *do not mutually touch*, so that they cannot have more than five points *arbitrarily chosen* in common.

Mr. Weddle sent a solution to this exercise.

LII. *Mr. Thomas Tate, Battersea Training Institution.*

If the formula, $V = \frac{n}{P} + m$, express the relation between the volume and pressure of steam, raised from a unit of water, and if U be put for the *work* performed by the expansion of a cubic foot of water, in the form of steam, between the pressures P and p lbs. per square inch; then

$$e^{\frac{U}{144n}} = \frac{P}{p}.$$

[First Solution. *By the Proposer.*]

It may be shown, generally, that the work performed by the expansion of a given volume of steam is not affected by the *form* of the space through which it is allowed to expand. Let us suppose, therefore, that the steam is enclosed in a cylinder whose piston is one inch in surface, and let x, in units of feet, be any position of the piston corresponding to r pressure, a the position when the pressure is P, and a_1 the position when the pressure is p; then we have

$$\text{the volume of steam at } r \text{ pressure} = \frac{x}{144};$$

consequently, $\frac{x}{144} = \frac{n}{r} + m$, and $r = \frac{1}{ax - b}$; where $a = \frac{1}{144n}$ and $b = \frac{m}{n}$. Similarly, we get

$$P = \frac{1}{aa - b}, \text{ and } p = \frac{1}{aa_1 - b};$$

$$\therefore U = \int_a^{a_1} r\,dx = \int_a^{a_1} \frac{dx}{ax - b} = \frac{1}{a} \log_e \frac{aa_1 - b}{aa - b}$$

$$= \frac{1}{a} \log_e \frac{P}{p} = 144n \log_e \frac{P}{p};$$

$$\therefore e^{\frac{U}{144n}} = \frac{P}{p}.$$

[Second Solution. *Mr. G. W. Hearn; and similarly by Mr. J. Reid.*]

The work performed by any machine is measured by the force multiplied by the space through which it acts. Let a be the area of the piston in square inches, x the distance of the piston from the bottom of the cylinder in feet; then $v = ax$, and work done $= \int apdx$. But since $v = \frac{n}{p} + m$;

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therefore $dv = a\,dx = -\frac{n\,dp}{p^2}$, and hence work done $= -\int\frac{n\,dp}{p}$. If then the unit of water in the formula $v = \frac{n}{p} + m$ be considered to be twelve cubic inches, the work done by a cubic foot of water, converted into steam, will be 144 times greater. Hence

$$U = -144n\int\frac{dp}{p} = -144n\log_e p + c.$$

Let P be the pressure of steam before the piston begins to ascend, so that when $p = P$, $U = 0$, and we have

$$U = 144n\log_e\frac{P}{p}, \text{ or } e^{\frac{U}{144n}} = \frac{P}{p}.$$

A solution was received from Mr. R. H. Wright, London.

LIII. *Mr. Robert Finlay, Professor of Mathematics, Manchester New College.*

If three conic sections have only one common secant (real or ideal), any transversal cuts them in six points which are in involution.

[Solution. *Mr. G. W. Hearn, Sandhurst.*]

Let $u = 0$ be an equation of the second order between x and y, $v = 0$ an equation of the first order between the same variables; then λ being an arbitrary multiplier,

$$u + \lambda v = 0,$$

will represent a system of conic sections having but one common secant $v = 0$. In this, putting $y = 0$, we will have an equation of the form

$$ax^2 + ex + 1 + \lambda(ax + 1) = 0;$$

and if in this we put $x + m$ for x, we have

$$x^2 + \left(2m + \frac{e + a\lambda}{a}\right)x + m^2 + m.\frac{e + a\lambda}{a} + \frac{1 + \lambda}{a} = 0;$$

therefore, if x_1, x_2 are the roots of this equation, we have the product of the roots

$$x_1x_2 = m^2 + m\frac{e + a\lambda}{a} + \frac{1 + \lambda}{a}.$$

If then this product can be independent of λ, it must remain unchanged when $\lambda = 0$;

$$\therefore\quad m^2 + m\frac{e + a\lambda}{a} + \frac{1 + \lambda}{a} = m^2 + m\frac{e}{a} + \frac{1}{a};$$

$$\therefore\quad \frac{ma\lambda}{a} + \frac{\lambda}{a} = 0, \text{ or } m = -\frac{1}{a}.$$

Hence the axis of x will be cut in involution by any number of such conic sections, the distance of the centre of involution from the origin being $= -\frac{1}{a}$. But the axis of x may be any line whatever; hence any transversal cuts the curves in involution.

The solutions by the Proposer and Mr. Weddle are unavoidably postponed; but should the subject be resumed in our next number, they will then appear, as they are well worthy of insertion.

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THE MATHEMATICIAN.

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PROPOSITIONS ON THE CONIC SECTIONS.

[*Mr. James Dalmahoy, Edinburgh.*]

(Continued from p. 294, vol. i.)

PROPOSITION VI. PROBLEM.

To divide a given Ellipse into five equal Sectors.

LEMMA I.

In a conic section ALBMC, *let two chords* BA, BC *be drawn from the same point* B, *and let them be cut by any parallel* EF *to the tangent at* B; *also let* DH, DK *be semi-diameters parallel to the chords* BA, BC; *then*

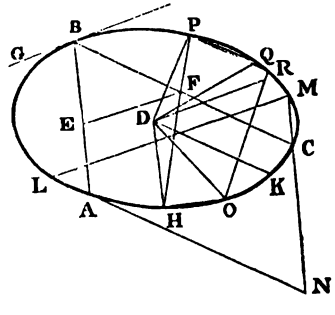
$$BA.BE : BC.BF :: DH^2 : DK^2.$$

Through A and C draw the chords AL and CM parallel respectively to BC and BA; produce them to meet in N, and join LM. Then it is evident from the familiar properties of the figure, that BA = NC, and BC = NA. Also, since

$$MN.NC : LN.NA :: DH^2 : DK^2$$

therefore,

$$MN.BA : LN.BC :: DH^2 : DK^2 \ldots (1)$$



But by Simson's Conic Sections, book iv, prop. xxix, cor. 2, the line LM is parallel to the tangent BG, and therefore it is parallel to EF; hence the triangles BEF and NML are evidently similar: therefore

$$BE : BF :: NM : NL \ldots\ldots\ldots\ldots\ldots (2)$$

Multiply together the corresponding terms of the analogies (1) and (2), and dividing the homologous terms by the quantity common to each, the resulting analogy is

$$BA.BE : BC.BF :: DH^2 : DK^2.$$

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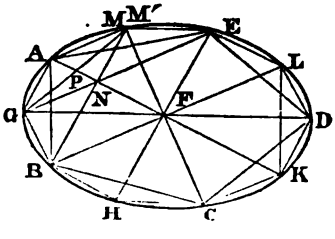
LEMMA II.

In the ellipse of which D *is the centre, let* HDO, PDQ *be two equal sectors, the segments* HOH, PQP *corresponding to these, are also equal.*

For if not, let the segment PRP be equal to HOH, and draw OR, PH, RD; then (Lemma to the Porism in the Mathematician vol. i. p. 123) the line RO is parallel to PH, and (Stone's Conic Sections, b. v. prop. iv.) the sectors HDO and PDR are equal, but by hypothesis the sectors HDO and PDQ are equal; therefore the sectors PDQ and PDR are equal, or the part is equal to the whole, which is absurd; hence the segment HOH is equal to the segment PQP.

ANALYSIS OF THE PROBLEM.

Let it be supposed that the ellipse, of which the centre is F, is divided into five equal sectors, by semi-diameters drawn to the points A, B, C, D, E; and let the adjacent points be joined, forming an inscribed pentagon. Produce each of the semi-diameters to meet the curve on the opposite side in the points K, L, M, G, H; and let the adjacent points be again joined, forming an inscribed decagon. Draw chords joining the points G, M, and G, E, the former intersecting the semi-diameter FA in the point P, the latter, in the point N: from B draw a line through N meeting the curve again in M'.



Since *(Lemma* 11*)* the segment AGB is equal to ELD, and the segment BHC is equal to DKC, the chord AE is parallel to a chord through B, D, which again is parallel to the tangent at C, and also to the tangent at M; hence the segments MA and ME, and also the corresponding sectors are equal. In the same manner it may be proved that the other four sectors are divided equally by the semi-diameters drawn to the points G, H, K, L; the ellipse is therefore divided into ten equal sectors by the five diameters AK, BL, CM, DG, EH.

Again, the segment KDL being equal to the opposite segment BGA, which by hypothesis is equal to AME, therefore KDL = AME, and hence the chord EL is parallel to AK; also, since the segment BG is equal to the segment LE, the chord GE is parallel to BL; hence FLEN is a parallelogram, and the line NE = FL = FB. But in the two triangles BFN and ENF, the side FB is equal to NE; the side FN is common, and the angle BFN is equal to the alternate angle ENF, therefore the triangles are equal, and the corresponding alternate angles BNF and EFN are equal; therefore BN or BM' is parallel to FE or HE; hence the segment HB = EM' = EM; consequently the points M and M' coincide.

It easily follows from what has been proved that the figure AGNM is a parallelogram, of which the side AG is parallel to the semi-diameter FE, and AM parallel to FB; also that GM is parallel to the tangent at A; hence *(Lemma* 1.*)* $KA.AP : AG^2 :: FA^2 : FE^2$ and $KA.AP : FA^2 :: AG^2 : FE^2$.

But the diagonals GM and AN divide each other equally in P; therefore KA.AP = AF.AN; also, from the properties of diverging lines cut by parallels, $AN^2 : NF^2 :: AG^2 : FE^2$, hence, by equality of ratios, $AF.AN : AF^2 :: AN^2 : NF^2$; therefore AN : AF :: $AN^2 : NF^2$; hence NF is a mean proportional between AN and AF, or the semi-diameter AF is divided

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at the point N in extreme and mean ratio. But the position of A being arbitrary, the semi-diameter FA is given, therefore the points N, P, and the chords GM, AG and AM are given. Also the semi-diameters FB, FC, FD and FE are given; hence the semi-diameters which divide the ellipse into five equal sectors are given in position.

COMPOSITION.

Assume A, any point of the curve, and draw the diameter AK: divide the semi-diameter FA in N in such manner that $AF.AN = NF^2$ and therefore $AN : AF :: AN^2 : NF^2$. Divide AN equally in P; through P draw the chord GM conjugate to AF, that is, parallel to the tangent at A: draw the diameters GD and MC; also the chords AG and AM, and parallel to these respectively draw the diameters HE and BL: the ellipse is divided into five equal sectors by the semi-diameters FA, FB, FC, FD, FE.

For join the points G, E and M, B. It is evident that $KA.AP = FA.AN$; therefore by Lemma 1. $AF.AN : AF^2 :: AG^2 : FE^2$, or $AN : AF :: AG^2 : FE^2$; but by hypothesis $AN : AF :: AN^2 : NF^2$; therefore $AN : NF :: AG : FE$. In the same manner it may be proved that $AN : NF :: AM : FB$; therefore the chords GE and MB intersect the semi-diameter FA, and each other, in the point N.

Further, in the triangles NPG and APM, the side NP is equal to AP and GP to MP; also the contained angle NPG is equal to APM; therefore the triangles are equal, and the corresponding alternate angles NGP, AMP are equal; hence the chords GE and AM are parallel, and the segment AG is equal to the segment ME. In the same manner, from the equality of the triangles NPM and APG, it may be proved that the chords BM and AG are parallel, and the segment AM equal to GB. Again, GE being parallel to AM, it is parallel to BL; therefore the segment $EL = GB = AM$; but the chord GM being parallel to the tangent at A, the segment $AG = AM$; consequently the five segments BG, GA, AM, ME, EL, and their corresponding sectors, are equal to one another, and as each of them is equal to its opposite sector included between the same two diameters, it follows that the ten sectors are equal to one another, and consequently that the ellipse is divided into five equal sectors by the alternate semi-diameters FA, FB, FC, FD, FE.

Scholium. It appears from the foregoing demonstration that the processes for dividing the circle and the ellipse into five equal sectors, are exactly similar: so also, it may be remarked, are those for their division into three equal sectors.

ON INVOLUTION.

[*Professor Hearn, Royal Military College, Sandhurst.*]

Let $u = 0$, $v = 0$, be two equations in x and y of the n^{th} order; then the equation

$$u + \lambda v = 0 \ldots\ldots \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

being satisfied by $u = 0$, $v = 0$, is the equation of a curve of the n^{th} order passing through all the points of intersection, real or ideal, of the two curves $u = 0$, $v = 0$.

Put $x + h$ instead of x, and $y + k$ instead of y, and then put $y = ax$; we then have for the coefficient of x^r in u,

$$\frac{1}{1.2.3..r}\left\{\frac{d^r u}{dh^r} + ra\frac{d^{r-1}u}{dh^{r-1}dk} + \frac{r(r-1)}{1.2}a^2.\frac{d^{r-2}u}{dh^{r-2}dk^2} + \ldots\frac{d^r u}{dk^r}\right\},$$

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which may be symbolized thus,

$$\frac{1}{1.2.3..r}\left\{\frac{d}{dh}+a\frac{d}{dk}\right\}^r \underline{u}$$

Similarly, the coefficient of x^r in v may be written

$$\frac{1}{1.2.3..r}\left\{\frac{d}{dh}+a\frac{d}{dk}\right\}^r \underline{v}$$

where $\underline{u}$ and $\underline{v}$ are u and v expressed in h and k instead of x and y respectively.

Hence, if A be the coefficient of x^n in u, and B that of x^n in v, the equation (1) may after the above changes be written

$$(A+\lambda B)\,x^n + \&c. \frac{1}{1.2.3..r}\left(\frac{d}{dh}+a\frac{d}{dk}\right)^r (\underline{u}+\lambda\underline{v}) \ldots = 0$$

Hence the sum of the products of the roots of this equation taken $n-r$ together will be

$$\frac{1}{1.2.3..r}\left(\frac{d}{dh}+a\frac{d}{dk}\right)^r (\underline{u}+\lambda\underline{v}) \div (A+\lambda B).$$

If this quantity is independent of λ it must remain the same when $\lambda = 0$. Denoting the operation $\frac{d}{dh}+a\frac{d}{dk}$ by Δ, we have, therefore,

$$\frac{\Delta^r(\underline{u}+\lambda\underline{v})}{A+\lambda B} = \frac{\Delta_r \underline{u}}{A};$$

$$\therefore\ A\,\Delta^r \underline{v} = B\,\Delta^r \underline{u}.$$

Hence, if this sum of the products of the roots taken r and r together be the same for all the curves, the locus of the centres of involution of the r^{th} order will be

$$\Delta^{n-r}(A\underline{v} - B\underline{u}) = 0;$$

in which we have again put x and y, instead of h and k, and Δ of course signifies $\frac{d}{dx}+a\frac{d}{dy}$.

Hence, if a system of curves of the n^{th} order be drawn through the n^2 intersections of two curves $u=0, v=0$ of the n^{th} order, and the system be cut by a series of transversals parallel to $y = ax$, a point may be found in each transversal such, that the distances of the points of intersection of any one of the curves with that transversal from the point found in it multiplied together r and r, and the sum of the products taken, shall be the same for all the curves; and the locus of such centres of involution will be the curve defined by the equation

$$\Delta^{n-r}(A\underline{v} - B\underline{u}) = 0.$$

By the same mode of reasoning if $u=0, v=0$ be two equations of the n^{th} order between $x, y, z,$; then $u+\lambda v = 0$ is the equation to a surface passing through the curves of mutual intersection of $u=0, v=0$; and if $y = ax$, $z = \beta x$ be the equations to a parallel to each of a system of transversals, we have for the locus of the centre of involution of the r^{th} order

$$\Delta^{n-r}(A\underline{v} - B\underline{u}) = 0,$$

where Δ means $\frac{d}{dx}+a\frac{d}{dy}+\beta\frac{d}{dz}$.

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It is not by any means essential to the above that u and v should be of the same dimensions, or in other words, either A or B may be zero.

When $r = n$, the above equations become simply $Av - Bu = 0$. This is the kind of involution usually understood by the writers on this subject.

As an illustration of the preceding, let

$$u = x^2 - ay = 0,$$
$$v = y^2 - ax = 0,$$

and let $y = x$ be the line parallel to each of the transversals.

In this case $A = 1$, $B = 1$, $a = 1$, and the locus of the centres of involution is

$$y^2 - x^2 - a(x - y) = 0,$$

or,
$$(y - x)\{y + x + a\} = 0;$$
$$\therefore\ y - x = 0, \text{ or } y + x + a = 0.$$

For the first of these the centre is infinitely distant, the second is therefore to be taken.

The equation to the resulting system of conic sections is

$$\lambda y^2 + x^2 - ay - a\lambda x = 0,$$

in which λ is arbitrary. Hence if $y = x + \beta$ be a secant cutting each of this system of conic sections, the product of the distances of the points of intersection of any one conic section with this secant, measured from the point $x = -\frac{a+\beta}{2}$, $y = -\frac{a-\beta}{2}$ will be the same as the like product for any other conic section of the system.

Cor. Let there be two equal parabolas having a common vertex V and axes at right angles; and let P be their other point of intersection. Join PV and bisect it in D, and in PV produced make VE = VD. Draw EL perpendicular to EV. Let LMNRS be a secant parallel to EP cutting one curve in M, S, and the other in N, R; then

$$LM.LS = LN.LR.$$

As another illustration, take the two cubical parabolas

$$u = x^3 - a^2y = 0 \qquad \therefore\ A = 1$$
$$v = y^3 - a^2x = 0 \qquad\quad B = a^3$$

and $y = ax$ for the direction of the secants.

Moreover, make $r = 2$; then

$$\frac{du}{dx} + a\frac{du}{dy} = 3x^2 - a^2a,$$

$$\frac{dv}{dx} + a\frac{dv}{dy} = -a^2 + 3ay^2;$$

$$\therefore\ -a^2 + 3ay^2 - a^3(3x^2 - a^2a) = 0,$$

$$\text{or}\qquad y^2 - a^3x^3 = \frac{a^2(1 - a^4)}{3a}.$$

Hence if from any point in this hyperbola a line parallel to $y = ax$ be drawn, cutting the system of curves of which

$$x^3 + \lambda y^3 - a^2(y + \lambda x) = 0$$

is the equation, the sum of the products, two and two, of the distances of the points of intersection of the curve from the assumed point in the hyperbola,

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will be the same for all individuals of the system. I will not multiply illustrations further than to notice one more.

When u and v are the equations of two conic sections which intersect each other in four points; then $u + \lambda v = 0$ may be regarded as the equation to a system of conic sections circumscribing a given quadrilateral. If then through any point of the hyperbola $Av - Bu = 0$, a line be drawn parallel to one of its asymptotes the product of the distances of the points of its intersection with any of the conic sections circumscribing the quadrilateral, from the assumed point of the hyperbola, will be the same for all individuals of the system.

What is here written seems to include all that others have said on the subject, and if I mistake not goes much further.

ON THE THEOREMS IN SPACE ANALOGOUS TO A CERTAIN PROPERTY OF CURVES OF THE SECOND ORDER.

[*Mr. Thomas Weddle, Newcastle.*]

It is a known property that,

(A) If a *diameter*, conjugate to a given *line*, be drawn to each of the indefinite number of curves of the second order that pass through *four* given points in a plane, all these diameters will intersect in the same *point*.

In the Lady's and Gentleman's Diary for 1843, a theorem which was probably considered the analogous property in space, was proposed for solution. It may be enunciated as follows:—

(B) If a *diameter*, conjugate to a given *plane*, be drawn to each of the indefinite number of surfaces of the second order that pass through *eight* given points in space, all these diameters will intersect in the same *point*.

This theorem is, however, true only in particular cases as has been shown by Mr. Fenwick at p. p. 12—14, of No. 1. Vol. II. of this work. Mr. F. remarks, "It is very probable, that many of those theorems in plano that might be supposed to have corresponding properties in space, in respect of lines meeting in a point, fail in a similar way as to the general case. In a plane, two lines *necessarily* intersect one another, either at a finite or an infinite distance. This, however, is not the case in space, for the lines may be in different planes." It must, however, be borne in mind that *planes* in space may correspond to *lines* in plano, and that two planes "necessarily intersect one another either at a finite or an infinite distance." Hence, should we find that any theorem in a plane in respect of intersecting lines is not true in space in respect of *lines*, it may be worth while to inquire whether it may not be true in respect of *planes*. Instead, then, of endeavouring to find the theorem in space which is analogous to (A), by retaining the *diameters*, and changing the given *line* into a given *plane*, let us replace the *diameters* by *diametral planes* and consequently retain the given *line*. Hence, observing that two planes intersect in a line, we shall have to try the following theorem.

(C) If a *diametral plane*, conjugate to a given line, be drawn to each of the indefinite number of surfaces of the second order that pass through *eight* given points in space, all these diametral planes will intersect in the same *line*.

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We proceed to ascertain whether or not this theorem is true.

Let $u = 0$, $v = 0$, be the equations to two of the surfaces that pass through the eight points, and $u' = 0$, $v' = 0$, their diametral planes, conjugate to the given line. All the surfaces of the second order passing through the same eight points, may obviously be represented by

$$u + \lambda v = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

λ being an arbitrary multiplier. That diametral plane of (1), which is conjugate to the given line, is readily found (Gregory's Solid Geometry, art. 96) to be

$$u' + \lambda\ v' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now (2) is satisfied, whatever λ may be, by $u' = 0$, and $v' = 0$, which denote two planes. The intersection of these planes will therefore determine a straight line through which all the diametral planes will pass. The theorem (C) is consequently established.

We may observe moreover, that while *two* planes intersect in a *line*, *three* planes intersect in a *point*. Hence if we can introduce another arbitrary multiplier into the general equation of the surfaces, so as to cause the equation of the diametral planes to separate into three equations of the first degree, we need not change the word "*point*" in (A) into "*line*," as has been done in order to get (C). Now such a multiplier may be introduced by supposing the surfaces to pass through *seven* given points only, and in this way the theorem analogous to (A) may be enunciated as follows :—

(D) If a *diametral plane*, conjugate to a given *line*, be drawn to each of the indefinite number of surfaces of the second order that pass through *seven* given points in space, all these diametral planes will intersect in the same *point*.

This theorem is easily established. Let $u = 0$, $v = 0$, $w = 0$, be three of the surfaces passing through the seven points; and $u' = 0$, $v' = 0$, $w' = 0$, their diametral planes conjugate to the given line. The equation of all the surfaces passing through the seven points may be denoted by

$$u + \lambda v + \mu w = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where λ and μ are arbitrary multipliers. The diametral plane of (3) conjugate to the given line is

$$u' + \lambda v' + \mu w' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

Now (4) is satisfied, whatever be λ and μ, by $u' = 0$, $v' = 0$, $w' = 0$, which denote three planes. The intersection of these planes will therefore determine a point, through which all the diametral planes will pass. Hence theorem (D) is true.

Each of the theorems (C), (D) is as strictly analogous to (A) as (B) is, and they are always true, whereas (B) is not so except in particular cases. From this we see that while some theorems in plano have perhaps none corresponding to them in space, others have more analogous properties than one.

Newcastle-upon-Tyne, Nov. 24th, 1845.

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SIMPLIFICATIONS IN ELEMENTARY TRIGONOMETRY.

[*From a Correspondent.*]

I. PLANE TRIGONOMETRY.

Denoting the sides of the triangle respectively opposite to the angles A, B, C by a, b, c in the usual manner, we have

(1) $$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= b^2 + c^2 + 2bc - 2bc - 2bc\cos A \\ &= (b+c)^2 - 2bc\,(1+\cos A) \\ &= (b+c)^2 - 4bc\,\cos^2\frac{A}{2}; \end{aligned}$$

$$\therefore\quad \cos^2\frac{A}{2} = \frac{(b+c)^2 - a^2}{4bc} = \frac{s\,(s-a)}{bc}.$$

(2) $$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= b^2 + c^2 - 2bc + 2bc - 2bc\cos A \\ &= (b-c)^2 + 2bc\,(1-\cos A) \\ &= (b-c)^2 + 4bc\sin^2\frac{A}{2}; \end{aligned}$$

$$\therefore\quad \sin^2\frac{A}{2} = \frac{a^2 - (b-c)^2}{4bc} = \frac{(s-b)\,(s-c)}{bc}.$$

(3) $$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= b^2 + c^2 + 2bc - 2bc - 2bc\cos A \\ &= (b+c)^2 - 4bc\cos^2\frac{A}{2} \\ &= (b+c)^2 - N^2 \qquad \left(\text{where } N = 2\cos\frac{A}{2}\sqrt{bc}\right); \end{aligned}$$

$$\therefore\quad a = \{(b+c+N)\,(b+c-N)\}^{\frac{1}{2}}.$$

a formula well adapted for logarithmic computation.

II. CORRESPONDING FORMULA IN SPHERICAL TRIGONOMETRY.

Using similar notation for the sides and angles of a spherical triangle, we have

(1) $$\begin{aligned} \cos a &= \cos b\cos c + \sin b\sin c\cos A \\ &= \cos b\cos c - \sin b\sin c + \sin b\sin c + \sin b\sin c\cos A \\ &= \cos(b+c) + \sin b\sin c\,(1+\cos A) \\ &= \cos(b+c) + 2\sin b\sin c\cos^2\frac{A}{2}; \end{aligned}$$

$$\therefore\quad \cos^2\frac{A}{2} = \frac{\cos a - \cos(b+c)}{2\sin b\sin c} = \frac{\sin s\,\sin(s-a)}{\sin b\sin c}.$$

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(2) $\cos a = \cos b \cos c + \sin b \sin c \cos A$
$= \cos b \cos c + \sin b \sin c - \sin b \sin c + \sin b \sin c \cos A$
$= \cos (b - c) - \sin b \sin c (1 - \cos A)$
$= \cos (b - c) - 2 \sin b \sin c \sin^2 \frac{A}{2}$;

$\therefore \sin^2 \frac{A}{2} = \frac{\cos (b - c) - \cos a}{2 \sin b \sin c} = \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}$.

(3) $\cos a = \cos b \cos c + \sin b \sin c \cos A$
$= \cos b \cos c - \sin b \sin c + \sin b \sin c + \sin b \sin c \cos A$
$= \cos (b + c) + \sin b \sin c (1 + \cos A)$
$= \cos (b + c) + 2 \sin b \sin c \cos^2 \frac{A}{2}$;

$\therefore 1 - \cos a = 1 - \cos (b + c) - 2 \sin b \sin c \cos^2 \frac{A}{2}$,

or $2 \sin^2 \frac{a}{2} = 2 \sin^2 \frac{b + c}{2} - 2 \sin b \sin c \cos^2 \frac{A}{2}$.

Dividing by 2, and putting $\sin^2 N = \sin b \sin c \cos^2 \frac{A}{2}$, we get

$$\sin^2 \frac{a}{2} = \sin^2 \frac{b + c}{2} - \sin^2 N = \left(\sin \frac{b + c}{2} + \sin N\right)\left(\sin \frac{b + c}{2} - \sin N\right).$$

Greenwich, November, 1845. J. R.

ON SOME RELATIONS BETWEEN THE RADII OF THE CIRCLES WHICH TOUCH THE CIRCLES DESCRIBED ON THE SIDES OF A PLANE TRIANGLE.

[*Mr. Rutherford.*]

In the first volume of the *Mathematician* several demonstrations are given of an elegant theorem with reference to the relation between the reciprocals of the radii of the eight circles which may in general be described to touch three given circles in a plane, and a particular case of the theorem was proposed for demonstration in the Lady's and Gentleman's Diary for 1845. The following proof of the property, involving very neat and symmetrical values of the reciprocals of the several radii of the circles of contact, was sent to the editor of that valuable periodical, but want of space unavoidably precluded its insertion in the Diary for the present year.

Our object in the subsequent paper will be the solution of the following problem, and thence to deduce some relations between the radii of the tangent circles.

PROBLEM.

To determine the values of the radii of the circles described to touch the circles whose diameters coincide in magnitude and position with the three sides of a plane triangle.

Let A, B, C be the centres of the three circles described on EF, FD, DE, the sides of the triangle DEF, and O the centre of the circle which touches the three other circles internally. Draw AB, BC, CA, OA, OB, OC, and let $BC = a$, $CA = b$, $AB = c$, $2s = a + b + c$; the radius of the

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circle inscribed in the triangle ABC $= r$, and adopt the usual notation with respect to the circumscribed and escribed circles. Let also s_1, s_2, s_3 be written for $s-a$, $s-b$, $s-c$, and let ρ_1 denote the radius of the circle centre O; then if OAB $= \theta$, and OAC $= \phi$, we have $\frac{\theta}{2}+\frac{\phi}{2}=\frac{A}{2}$*, and by taking the cosines of both members of this angular equality there arises the condition

$$\sin^2\frac{A}{2}-\cos^2\frac{\theta}{2}-\cos^2\frac{\phi}{2}+2\cos\frac{A}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2}=0\ldots(1)$$

Now

$$\cos\theta=\frac{c^2+(a-\rho_1)^2-(b-\rho_1)^2}{2c(a-\rho_1)};\ 1+\cos\theta=\frac{(a+c-\rho_1)^2-(b-\rho_1)^2}{2c(a-\rho_1)},$$

$$\text{or } 2\cos^2\frac{\theta}{2}=\frac{(a-b+c)(a+b+c-2\rho_1)}{2c(a-\rho_1)}=\frac{2(s-b)(s-\rho_1)}{c(a-\rho_1)};$$

therefore,

$$\cos^2\frac{\theta}{2}=\frac{s_2}{c}\cdot\frac{s-\rho_1}{a-\rho_1}\quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Similarly

$$\cos^2\frac{\phi}{2}=\frac{s_3}{b}\cdot\frac{s-\rho_1}{a-\rho_1}\quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Hence $\cos^2\frac{\theta}{2}\cdot\cos^2\frac{\phi}{2}=\frac{s_2s_3}{bc}\left(\frac{s-\rho_1}{a-\rho_1}\right)^2=\sin^2\frac{A}{2}\cdot\left(\frac{s-\rho_1}{a-\rho_1}\right)^2$;

and therefore,

$$\cos\frac{\theta}{2}\cdot\cos\frac{\phi}{2}=\pm\sin\frac{A}{2}\cdot\frac{s-\rho_1}{a-\rho_1}\quad\ldots\ldots\ldots\ldots(4)$$

Substituting (2), (3), (4) in (1); transposing, and recollecting that $\sin^2\frac{A}{2}=-\frac{s_2s_3}{bc}$, we readily obtain

$$\sin^2\frac{A}{2}=\left(\frac{s_2}{c}+\frac{s_3}{b}\mp\sin A\right)\cdot\frac{s-\rho_1}{a-\rho_1}$$

$$\text{or, } \frac{a-\rho_1}{s-\rho_1}=\operatorname{cosec}^2\frac{A}{2}\left(\frac{s_2}{c}+\frac{s_3}{b}\right)\mp2\cot\frac{A}{2}=\frac{b}{s_3}+\frac{c}{s_2}\mp2\cot\frac{A}{2}\ldots(5)$$

Again, since $a=s_2+s_3$, $b=s_1+s_3$, $c=s_1+s_2$, and $\cot\frac{A}{2}=\frac{s_1}{r}$, the last equation becomes

$$\frac{a-\rho_1}{s-\rho_1}=\frac{s_1}{s_3}+\frac{s_1}{s_2}+2\mp\frac{2s_1}{r};$$

and subtracting each member from unity, and dividing by s_1, gives

$$\frac{1}{s-\rho_1}=\pm\frac{2}{r}-\frac{1}{s_1}-\frac{1}{s_2}-\frac{1}{s_3}\quad\ldots\ldots\ldots(6)$$

Now $rs=r_1s_1=r_2s_2=r_3s_3$, and $2D=-r+r_1+r_2+r_3$; therefore

$$\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}=\frac{r_1+r_2+r_3}{rs}=\frac{2D+r}{rs};$$

* The reader will readily draw the figure, and in this investigation the line AO is between the lines AB and AC; but whatever be the position of the centre O, the method of procedure, in each case, is precisely similar to that here pursued.

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hence (6) is changed to $\frac{1}{s-\rho_1} = \pm\frac{2}{r} - \frac{2D+r}{rs}$, from which we get,

$$\frac{s}{\rho_1} = 1 + \frac{r}{2(\pm s - D - r)} \quad \ldots\ldots\ldots\ldots (7)$$

the sign + giving the radius of the circle touching the three given ones *internally*, and the sign — that of the circle touching the same three circles *externally*. Hence if we denote the radius of the latter circle by ρ_8, we shall have

$$\frac{s}{\rho_1} = 1 + \frac{r}{2(s-D-r)}; \quad \frac{s}{\rho_8} = 1 - \frac{r}{2(s+D+r)}.$$

It will be convenient to designate the circles radii ρ_1, ρ_8 as the *first pair* of circles of contact; the circles radii ρ_1, ρ_7 which respectively touch the circle radius a externally and internally, and the two others internally and externally, the *second pair*: ρ_3, ρ_6 those touching the circle radius b only externally and internally, the *third pair*; and ρ_4, ρ_5 those touching only the circle radius c externally and internally, the *fourth pair*. Then by processes analogous to the preceding we shall have the following resulting expressions for the four pairs of radii of the tangent circles.

$$\left.\begin{aligned}
\frac{1}{s-\rho_1} &= \frac{2}{r} - \frac{1}{s_1} - \frac{1}{s_2} - \frac{1}{s_3} \\
\frac{1}{s_1-\rho_2} &= \frac{2}{r_1} - \frac{1}{s} + \frac{1}{s_2} + \frac{1}{s_3} \\
\frac{1}{s_2-\rho_3} &= \frac{2}{r_2} - \frac{1}{s} + \frac{1}{s_1} + \frac{1}{s_3} \\
\frac{1}{s_3-\rho_4} &= \frac{2}{r_3} - \frac{1}{s} + \frac{1}{s_1} + \frac{1}{s_2}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (A)$$

$$\left.\begin{aligned}
\frac{1}{s_3+\rho_5} &= -\frac{2}{r_3} - \frac{1}{s} + \frac{1}{s_1} + \frac{1}{s_2} \\
\frac{1}{s_2+\rho_6} &= -\frac{2}{r_2} - \frac{1}{s} + \frac{1}{s_1} + \frac{1}{s_3} \\
\frac{1}{s_1+\rho_7} &= -\frac{2}{r_1} - \frac{1}{s} + \frac{1}{s_2} + \frac{1}{s_3} \\
\frac{1}{s-\rho_8} &= -\frac{2}{r} - \frac{1}{s_1} - \frac{1}{s_2} - \frac{1}{s_3}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (B)$$

or,

(C)

$$\frac{1}{\rho_1} = \frac{1}{s} + \frac{r}{2s(s-D-r)}$$
$$\frac{1}{\rho_2} = \frac{1}{s_1} + \frac{r_1}{2s_1(s_1+D-r_1)}$$
$$\frac{1}{\rho_3} = \frac{1}{s_2} + \frac{r_2}{2s_2(s_2+D-r_2)}$$
$$\frac{1}{\rho_4} = \frac{1}{s_3} + \frac{r_3}{2s_3(s_3+D-r_3)}$$

(D)

$$\frac{1}{\rho_8} = \frac{1}{s} - \frac{r}{2s(s+D+r)}$$
$$\frac{1}{\rho_7} = -\frac{1}{s_1} + \frac{r_1}{2s_1(s_1-D+r_1)}$$
$$\frac{1}{\rho_6} = -\frac{1}{s_2} + \frac{r_2}{2s_2(s_2-D+r_2)}$$
$$\frac{1}{\rho_5} = -\frac{1}{s_3} + \frac{r_3}{2s_3(s_3-D+r_3)}$$

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Thus we have investigated simple and symmetrical expressions for the values of either the reciprocals of the radii, or the radii themselves, of the eight tangent circles. We shall now deduce some relations between these radii or their reciprocals.

The radii of the tangent circles being functions of the three sides of the triangle ABC, it is obvious that if the quantities a, b, c, be eliminated from the *eight* equations (A), (B), or (C), (D), we shall have five resulting equations involving simply the eight radii of the tangent circles, and consequently there are *five* independent relations between the several radii. The elimination of a, b, c from these equations will be attended with considerable difficulty; but there are two instances in which it can easily be effected, and these are given below.

It will be necessary, however, first to show that

$$s^2-(D+r)^2=s_1^2-(D-r_1)^2=s_2^2-(D-r_2)^2=s_3^2-(D-r_3)^2.$$

Since $s^2-s_1^2=(s+s_1)(s-s_1)=a(b+c)=ab+ac$

$=(r+r_1)(r_3+r_3)$ (*Lady's Diary* for 1843, p. 86.)

$=(r+r_1)(2D+r-r_1)$, since $2D=-r+r_1+r_2+r_3$

$=(D+r)^2-(D-r_1)^2$:

therefore $s^2-(D+r)^2=s_1^2-(D-r_1)^2$. In a similar manner the other equations are established.

Now putting $s^2-(D+r)^2=Q$, we have from the four pairs of equations in (C), (D)

$$\text{(1st pair)}\ldots\frac{1}{\rho_1}-\frac{1}{\rho_8}=\frac{r}{2s}\left\{\frac{1}{s-(D+r)}+\frac{1}{s+(D+r)}\right\}=\frac{r}{Q}$$

$$\text{(2nd pair)}\ldots\frac{1}{\rho_2}+\frac{1}{\rho_7}=\frac{r_1}{2s_1}\left\{\frac{1}{s_1+(D-r_1)}+\frac{1}{s_1-(D-r_1)}\right\}=\frac{r_1}{Q}$$

$$\text{(3rd pair)}\ldots\frac{1}{\rho_3}+\frac{1}{\rho_6}=\frac{r_2}{2s_2}\left\{\frac{1}{s_2+(D-r_2)}+\frac{1}{s_2-(D-r_2)}\right\}=\frac{r_2}{Q}$$

$$\text{(4th pair)}\ldots\frac{1}{\rho_4}+\frac{1}{\rho_5}=\frac{r_3}{2s_3}\left\{\frac{1}{s_3+(D-r_3)}+\frac{1}{s_3-(D-r_3)}\right\}=\frac{r_3}{Q}$$

Taking the reciprocals of these four equations, and recollecting that $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$, we have the relation

$$\frac{1}{\frac{1}{\rho_1}-\frac{1}{\rho_8}}=\frac{1}{\frac{1}{\rho_2}+\frac{1}{\rho_7}}+\frac{1}{\frac{1}{\rho_3}+\frac{1}{\rho_6}}+\frac{1}{\frac{1}{\rho_4}+\frac{1}{\rho_5}},$$

or $$\frac{\rho_1\rho_8}{\rho_8-\rho_1}=\frac{\rho_2\rho_7}{\rho_2+\rho_7}+\frac{\rho_3\rho_6}{\rho_3+\rho_6}+\frac{\rho_4\rho_5}{\rho_4+\rho_5}\ldots\ldots\ldots\ldots\text{(E)}$$

Again from the same equations (C), (D), we have

$$rs\left(\frac{1}{\rho_1}+\frac{1}{\rho_8}\right)=2r+\frac{r^2(D+r)}{Q};$$

$$r_1s_1\left(\frac{1}{\rho_7}-\frac{1}{\rho_2}\right)=-2r_1+\frac{r_1^2(D-r_1)}{Q}$$

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$$r_2 s_2\left(\frac{1}{\rho_6}-\frac{1}{\rho}\right)=-2r_2+\frac{r^2{}_2(D-r_2)}{Q};$$

$$r_3 s_3\left(\frac{1}{\rho_5}-\frac{1}{\rho_4}\right)=-2r_3+\frac{r^2{}_3(D-r_3)}{Q}$$

But $rs = r_1 s_1 = r_2 s_2 = r_3 s_3 = \Delta$, and taking the sum of these, we get

$$\Delta\left(\frac{1}{\rho_1}+\frac{1}{\rho_8}+\frac{1}{\rho_7}-\frac{1}{\rho_2}+\frac{1}{\rho_6}-\frac{1}{\rho_3}+\frac{1}{\rho_5}-\frac{1}{\rho_4}\right)$$
$$=-4D+\frac{(r^2+r^2{}_1+r^2{}_2+r^2{}_3)D+r^3-r^3{}_1-r^3{}_2-r^3{}_3}{Q}.$$

ReplacingDand Q by their values $\frac{1}{2}(-r+r_1+r_2+r_3)$ and $s^2-(D+r)^2$, it will be found that the fractional part of the member on the right of this equation is $= 4D$; hence we have by transposition, another beautiful relation,—viz.

$$\frac{1}{\rho_2}+\frac{1}{\rho_3}+\frac{1}{\rho_4}=\frac{1}{\rho_1}+\frac{1}{\rho_5}+\frac{1}{\rho_6}+\frac{1}{\rho_7}+\frac{1}{\rho_8}\quad\ldots\ldots(F)$$

These two relations (E), (F) between the radii of the eight tangent circles may be thus enunciated.

(E). *If circles be described on the three sides of a plane triangle, as diameters, and circles be described to touch them, twice the product of the radii of the circles which touch the three given circles internally and externally divided by the difference of their radii, is equal to the sum of the harmonic means between the radii of the three pairs of circles which touch only one of the given circles internally and externally.*

(F). *If circles be described on the three sides of a plane triangle as diameters, and circles be described to touch them, the sum of the reciprocals of the radii of the three circles which touch only one of the given circles externally, is equal to the sum of the reciprocals of the radii of the other five.*

The investigations of these two neat relations have offered little difficulty; but the complete elimination of the sides of the triangle from the eight equation (C, D), does not appear to be easily effected. If, however, the remaining three relations should be communicated to us we would avail ourselves of an early opportunity of placing them on record.

Cor. 1. When the angle D of the original triangle DEF is a right angle, the angle A of the triangle ABC is also a right angle; hence, in this case, $a^2=b^2+c^2$, $D=a$, $r=s_1=s-a$, $2\Delta=bc$, $r_1 s_1=r_2 s_2=r_3 s_3=\frac{1}{2}bc$; consequently the modified values of the expressions for the reciprocals of the radii, are

$$\rho_1=0,\quad \rho_2=0,\qquad \rho_5=0,\quad \rho_6=0,$$

$$\frac{1}{\rho_3}=\frac{1}{s_2}+\frac{r_3}{2s_3(s_2+a-r_2)},\qquad \frac{1}{\rho_7}=-\frac{1}{s_1}+\frac{r_1}{2s_1(s_1-a+r_1)},$$

$$\frac{1}{\rho_4}=\frac{1}{s_3}+\frac{r_3}{2s_3(s_3+a-r_3)},\qquad \frac{1}{\rho_8}=\frac{1}{s}-\frac{r}{2s(s+a+r)}.$$

Cor. 2. When the original triangle is equilateral, the triangle ABC is likewise equilateral; and, in this case, we have $a=b=c$, $s=\frac{3a}{2}$,

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$s_1=s_2=s_3=\frac{a}{2}$, $D=\frac{2a}{3}\sqrt{3}$, $r=\frac{a}{6}\sqrt{3}$, $r_1=r_2=r_3=\frac{a}{2}\sqrt{3}$, and the values of the reciprocals of the radii are as below:

$$\frac{1}{\rho_1}=\frac{3+\sqrt{3}}{2a},\quad \frac{1}{\rho_2}=\frac{1}{\rho_3}=\frac{1}{\rho_4}=\frac{3\sqrt{3}+1}{2a},$$

$$\frac{1}{\rho_8}=\frac{3-\sqrt{3}}{2a},\quad \frac{1}{\rho_5}=\frac{1}{\rho_6}=\frac{1}{\rho_7}=\frac{3\sqrt{3}-1}{2a}.$$

Cor. 3. On the sides of an equilateral triangle, as diameters, let circles be described, and if two circles be described to touch them internally and externally, the sum of the radii of these two circles is equal to a side of the triangle.

For, by *Cor.* 2, we have

$$\rho_1=\frac{2a}{3+\sqrt{3}}=\frac{a}{3}(3-\sqrt{3});\ \rho_8=\frac{2a}{3-\sqrt{3}}=\frac{a}{3}(3+\sqrt{3})$$

$$\therefore\ \rho_1+\rho_8=2a=\text{a side of the equilateral triangle DEF.}$$

NOTE ON A NEW GEOMETRICAL THEOREM.

[*The Rev. Charles Graves, A.M., Professor of Mathematics in the University of Dublin.*]

Propositions VII and XVI of Mr. Robert Finlay's elegant paper on the "Application of Algebra to the Modern Geometry" in *The Mathematician*, vol. i. pages 261 and 295, might have been proved with greater ease if he had previously established the following general Lemma, which is suggested by his demonstrations, if not virtually contained in them.

If $\phi(x,y)=ax^n+bx^{n-1}y+etc....+1=0$ *be the equation of a curve of the* n^{th} *degree, which is cut in the n points* $Q_1, Q_2, Q_3, ..Q_n$, *by a right line drawn from the origin* A *to the point* P *whose co-ordinates are* x', y', *we shall always have*

$$\phi(x',y')=\frac{PQ_1.PQ_2.PQ_3..PQ_n}{AQ_1.AQ_2.AQ_3..AQ_n}$$

This Lemma may be easily proved thus. Using p to denote the continued product of the distances of the origin from the n points in which the curve meets the axis of x, we shall have $p=\frac{1}{a}$. Now let us remove the origin of co-ordinates to the point x', y', by putting $x+x'$ and $y+y'$ in place of x and y; the absolute term in the transformed equation will be $\phi(x',y')$, and the coefficient of x^n will remain the same as before. We shall, therefore, have p', the continued product of the distances of the new origin from the n points in which the curve meets the new axis of x, $=\frac{\phi(x',y')}{a}$. Now it is a well known property of algebraic curves, that

$$\frac{p'}{p}=\frac{PQ_1.PQ_2.PQ_3..PQ_n}{AQ_1.AQ_2.AQ_3..AQ_n}.$$

Hence, it is proved that

$$\phi(x',y')=\frac{PQ_1.PQ_2.PQ_3..PQ_n}{AQ_1.AQ_2.AQ_3..AQ_n}.$$

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CONNECTION BETWEEN THE FORMULÆ IN PLANE AND SPHERICAL TRIGONOMETRY.

In any spherical triangle whose angles are A, B, C, and sides a, b, c,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad \ldots\ldots\ldots\ldots\ldots\ldots (A)$$

Differentiating, the angle A being constant, we have

$$-da \sin a = -db \sin b \cos c - dc \sin c \cos b + (db \cos b \sin c + dc \cos c \sin b) \cos A.$$

Now as a, b, c diminish, the triangle approaches to a plane triangle; the ultimate values of $\cos a$, $\cos b$, $\cos c$ are unity, and those of $\sin a$, $\sin b$, $\sin c$, a, b, c, da, db, and dc tend to equality. Hence for da, db, dc and $\sin a$, $\sin b$, $\sin c$, writing a, b, c; and for $\cos a$, $\cos b$, $\cos c$, writing unity, we have

$$-a^2 = -b^2 - c^2 + 2bc \cos A$$

$$\text{or} \quad a^2 = b^2 + c^2 - 2bc \cos A \quad \ldots\ldots\ldots\ldots\ldots\ldots (B)$$

(A) and (B) are the most general formulæ in plane and spherical trigonometry.

J. R.

Greenwich, November, 1845.

ON THE REDUCTION OF CERTAIN DEFINITE INTEGRALS TO MORE SIMPLE FORMS.

[*The Rev. Brice Bronwin, Penistone, Wakefield.*]

(Continued from p. 307, vol. i.)

In a former paper (vol. i. p. 303) I have given many examples illustrative of a simple and elegant method of reducing certain integrals to more simple ones. The present may be regarded as a continuation of the subject, the integrals here being of a more complex and general character.

Let $\Sigma_\lambda f(x) = f(x) + f(x + 2\lambda)$; then if $f(x) = \cos zx$, we have

$$\Sigma_\lambda \cos zx = \cos zx + \cos(z + 2\lambda)x = 2 \cos\lambda x \cos(z+\lambda)x$$

$$\Sigma_\lambda^2 \cos zx = 2^2 (\cos\lambda x)^2 \cos(z+2\lambda)x$$

$$\Sigma_\lambda^l \cos zx = 2^l (\cos\lambda x)^l \cos(z+l\lambda)x.$$

And therefore also,

$$\Sigma_\lambda^l \Sigma_\mu^m \cos zx = 2^{l+m} (\cos\lambda x)^l (\cos\mu x)^m \cos(z+l\lambda+m\mu)x.$$

Hence,

$$\int dx f(x) (\cos\lambda x)^l (\cos\mu x)^m \ldots \cos(z+l\lambda+m\mu+\ldots)x$$

$$= 2^{-l-m\ldots} \Sigma_\lambda^l \Sigma_\mu^m \ldots \int dx f(x) \cos zx \quad \ldots\ldots\ldots (A)$$

From this general formula many particular integrals, both definite and indefinite, may be derived.

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Make $f(x)=\frac{1}{h^2+x^2}$, then $\int_0^\infty \frac{dx\cos zx}{h^2+x^2}=\frac{\pi}{2h}c^{-hz}$; consequently,

$$\int_0^\infty \frac{dx\,(\cos\lambda x)^l\,(\cos\mu x)^m\,..\cos(z+l\lambda+m\mu..)x}{h^2+x^2}$$
$$=\frac{\pi c^{-hz}}{2h}\left(\frac{1+c^{-2h\lambda}}{2}\right)^l\left(\frac{1+c^{-2h\mu}}{2}\right)^m \ldots\ldots\ldots(1)$$

Here z must be positive. Differentiate for z, and we have

$$\int_0^\infty \frac{xdx(\cos\lambda x)^l\,(\cos\mu x)^m\,..\,\sin(z+l\lambda+m\mu....)x}{h^2+x^2}$$
$$=\frac{\pi c^{-hz}}{2}\left(\frac{1+c^{-2h\lambda}}{2}\right)^l\left(\frac{1+c^{-2h\mu}}{2}\right)^m \ldots.(2)$$

By changing h into $h\sqrt{-1}$, we obtain

$$\int_0^\infty \frac{dx(\cos\lambda x)^l\,(\cos\mu x)^m\,....\cos(z+l\lambda+m\mu....)x}{h^2-x^2}$$
$$=\frac{\pi}{2h}(\cos h\lambda)^l\,(\cos h\mu)^m....\sin(z+l\lambda+m\mu)h....(3)$$

$$\int_0^\infty \frac{xdx(\cos\lambda x)^l\,(\cos\mu x)^m....\sin(z+l\lambda+m\mu....)x}{h^2-x^2}$$
$$=-\frac{\pi}{2}(\cos h\lambda)^l\,(\cos h\mu)^m..\cos(z+l\lambda+m\mu..)h\ldots\ldots(4)$$

Since $\int_0^{\frac{\pi}{2}} \frac{dx\cos zx}{1-2p\cos 2x+p^2}=\frac{\pi}{2}\frac{p^{\frac{z}{2}}}{1-p^2}$, $p<1$; we find also from (A)

$$\int_0^{\frac{\pi}{2}} \frac{dx\,(\cos\lambda x)^l\,(\cos\mu x)^m\,..\cos(z+l\lambda+m\mu..)x}{1-2p\cos 2x+p^2}$$
$$=\frac{\pi}{2}\frac{p^{\frac{z}{2}}}{1-p^2}\left(\frac{1+p^\lambda}{2}\right)^l\left(\frac{1+p^\mu}{2}\right)^m \ldots\ldots(5)$$

Here z must be positive. Differentiating relative to it,

$$\int_0^{\frac{\pi}{2}} \frac{xdx\,(\cos\lambda x)^l\,(\cos\mu x)^m....\sin(z+l\lambda+m\mu....)x}{1-2p\cos 2x+p^2}$$
$$=-\frac{\pi}{4}\log p\,\frac{p^{\frac{z}{2}}}{1-p^2}\left(\frac{1+p^\lambda}{2}\right)^l\left(\frac{1+p^\mu}{2}\right)^m \ldots\ldots(6)$$

The preceding integrals have been obtained upon the supposition that l, m, *etc.*, are integers. But they are equally true, whatever be these quantities; for this follows immediately from the formula,

$$\phi(k)=\phi(o)+\frac{k}{1}\Delta\phi(o)+\frac{k(k-1)}{1\cdot 2}\Delta^2\phi(o)+etc.$$

Let $\Delta_\lambda z=2\lambda$, $\Delta_\mu z=2\mu$, *etc*; $\frac{d}{dz}=\mathrm{D}$. Then $\Delta_\lambda\cos zx=\cos(z+2\lambda)x-\cos zx=-2\sin\lambda x\sin(z+\lambda)x$, $\mathrm{D}\Delta_\lambda\cos zx=-2x\sin\lambda x\cos(z+\lambda)x$.

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By repeating the operations $D^l\Delta^l_\lambda \cos zx = (-2x)^l (\sin \lambda x)^l \cos(z + l\lambda)x$.

By operating in like manner for μ

$$D^{l+m}\Delta^l_\lambda \Delta^m_\mu \cos zx = (-2x)^{l+m}(\sin\lambda x)^l(\sin\mu x)^m \cos(z + l\lambda + m\mu)x.$$

Therefore,

$$\int dx\, x^{l+m}\cdots f(x)(\sin\lambda x)^l(\sin\mu x)^m \ldots \cos(z + l\lambda + m\mu \ldots)x$$
$$= \left(\frac{-1}{2}\right)^{l+m\ldots} D^{l+m\ldots}\Delta^l_\lambda \Delta^m_\mu \int dx f(x) \cos zx \ldots\ldots (B)$$

By putting for $f(x)$ the same functions as before, we find in the same manner

$$\int_0^\infty \frac{dx x^{l+m\ldots}(\sin\lambda x)^l(\sin\mu x)^m \ldots \cos(z+l\lambda+m\mu\ldots)x}{h^2 + x^2}$$
$$= \frac{\pi}{2h} h^{l+m\ldots} c^{-hz}\left(\frac{c^{-2h\lambda}-1}{2}\right)^l\left(\frac{c^{-2h\mu}-1}{2}\right)^m \ldots\ldots (7)$$

$$\int_0^\infty \frac{dx x^{l+l+m\ldots}(\sin\lambda x)^l(\sin\mu x)^m \ldots \sin(z+l\lambda+m\mu\ldots)x}{h^2 + x^2}$$
$$= \frac{\pi}{2} h^{l+m\ldots} c^{-hz}\left(\frac{c^{-2h\lambda}-1}{2}\right)^l\left(\frac{c^{-2h\mu}-1}{2}\right)^m \ldots\ldots (8)$$

$$\int_0^\infty \frac{dx x^{l+m\ldots}(\sin\lambda x)^l(\sin\mu x)^m \ldots \cos(z+l\lambda+m\mu\ldots)x}{h^2 - x^2}$$
$$= \frac{\pi h^{l+m\ldots}}{2h}(\sin h\lambda)^l(\sin h\mu)^m \ldots \sin(z+l\lambda+m\mu\ldots)h \ldots\ldots (9)$$

$$\int_0^\infty \frac{dx x^{l+l+m\ldots}(\sin\lambda x)^l(\sin\mu x)^m \ldots \sin(z+l\lambda+m\mu\ldots)x}{c^2 - x^2}$$
$$= -\frac{\pi}{h} h^{l+m\ldots}(\sin h\lambda)^l(\sin h\mu)^m \ldots \cos(z+l\lambda+m\mu\ldots)h \ldots (10)$$

$$\int_0^{\frac{\pi}{2}} \frac{dx x^{l+m\ldots}(\sin\lambda x)^l(\sin\mu x)^m \ldots \cos(z+l\lambda+m\mu\ldots)x}{1 - 2p\cos 2x + p^2}$$
$$= \frac{\pi(\log p)^{l+m\ldots}}{2} \frac{p^{\frac{z}{2}}}{1-p^z}\left(\frac{1-p^\lambda}{4}\right)^l\left(\frac{1-p^\mu}{4}\right)^m \ldots\ldots (11)$$

$$\int_0^{\frac{\pi}{2}} \frac{dx x^{l+l+m\ldots}(\sin\lambda x)^l(\sin\mu x)^m \ldots \sin(z+l\lambda+m\mu\ldots)x}{1 - 2p\cos 2x + p^2}$$
$$= -\frac{\pi}{4}(\log p)^{l+l+m\ldots} \frac{p^{\frac{z}{2}}}{1-p^z}\left(\frac{1-p^\lambda}{4}\right)^l\left(\frac{1-p^\mu}{4}\right)^m \ldots\ldots (12)$$

Let $\Sigma_{(\lambda,n)} f(z) = f(z) + f(z+2\lambda) + f(z+4\lambda) \ldots\ldots + f(z+2n\lambda)$.

Then

$$\Sigma_{(\lambda,n)} \cos zx = \cos zx + \cos(z+2\lambda)x \ldots\ldots + \cos(z+2n\lambda)x$$
$$= \cos(z+n\lambda)x \frac{\sin(n+1)\lambda x}{\sin\lambda x}.$$

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By repeating the operations, and operating for μ in like manner ;

$$\Sigma^{l}_{(\lambda,n)} \cos zx = \cos(z+nl\lambda)x \left\{\frac{\sin(n+1)\lambda x}{\sin\lambda x}\right\}^{l}; \quad \Sigma^{l}_{(\lambda,n)}\Sigma^{m}_{(\mu,p)}\cos zx$$
$$= \cos(z+nl\lambda+pm\mu)x \left\{\frac{\sin(n+1)\lambda x}{\sin\lambda x}\right\}^{l}\left\{\frac{\sin(p+1)\mu x}{\sin\mu x}\right\}^{m}.$$

Therefore,

$$\int dx f(x)\cos(z+nl\lambda+pm\mu..)x\left\{\frac{\sin(n+1)\lambda x}{\sin\lambda x}\right\}^{l}\left\{\frac{\sin(p+1)\mu x}{\sin\mu x}\right\}^{m}..$$
$$= \Sigma^{l}_{(\lambda,n)}\Sigma^{m}_{(\mu,p)}..\int dx f(x)\cos zx....(C)$$

Particularizing $f(x)$ as before, and observing that

$$\Sigma_{(\lambda,n)} c^{-hz} = c^{-hz}\left\{1+c^{-2h\lambda}+c^{-4h\lambda}+..+c^{-2nh\lambda}\right\}$$
$$= c^{-hz}\left(\frac{1-c^{-(n+1)2h\lambda}}{1-c^{-2h\lambda}}\right),$$

$$\Sigma_{(\lambda,n)} r^{\frac{z}{2}} = r^{\frac{z}{2}}\{1+r^{\lambda}+r^{2\lambda}+..r^{n\lambda}\} = r^{\frac{z}{2}}\left(\frac{1-r^{(n+1)\lambda}}{1-r^{\lambda}}\right).$$

Repeating the operations on $c^{-hz}, r^{\frac{z}{2}}$, we find

$$\int_0^{\infty}\frac{dx\cos(z+nl\lambda+pm\mu..)x}{h^2+x^2}\left\{\frac{\sin(n+1)\lambda x}{\sin\lambda x}\right\}^{l}\left\{\frac{\sin(p+1)\mu x}{\sin\mu x}\right\}^{m}....$$
$$= \frac{\pi c^{-hz}}{2h}\left\{\frac{1-c^{-(n+1)2h\lambda}}{1-c^{-2h\lambda}}\right\}^{l}\left\{\frac{1-c^{-(p+1)2h\mu}}{1-c^{-2h\mu}}\right\}^{m}..(13)$$

$$\int_0^{\frac{\pi}{2}}\frac{dx\cos(z+nl\lambda+pm\mu..)x}{1-2r\cos 2x+r^2}\left\{\frac{\sin(n+)1\lambda x}{\sin\lambda x}\right\}^{l}\left\{\frac{\sin(p+1)\mu x}{\sin\mu x}\right\}^{m}..$$
$$= \frac{\pi}{2}\frac{r^{\frac{z}{2}}}{1-r^2}\left\{\frac{1-r^{(n+1)\lambda}}{1-r^{\lambda}}\right\}^{l}\left\{\frac{1-r^{(p+1)\mu}}{1-r^{\mu}}\right\}^{m}.....(14)$$

Changing h into $h\sqrt{-1}$, and differentiating relative to z ;

$$\int_0^{\infty}\frac{dx\cos(z+nl\lambda+pm\mu..)x}{h^2-x^2}\left\{\frac{\sin(n+1)\lambda x}{\sin\lambda x}\right\}^{l}\left\{\frac{\sin(p+1)\mu x}{\sin\mu x}\right\}^{m}..$$
$$= \frac{\pi}{2h}\sin(z+nl\lambda+pm\mu..)h\left\{\frac{\sin(n+1)h\lambda}{\sin h\lambda}\right\}^{l}\left\{\frac{\sin(p+1)h\mu}{\sin h\mu}\right\}^{m}..(15)$$

$$\int_0^{\infty}\frac{xdx\sin(z+nl\lambda+pm\mu..)x}{h^2+x^2}\left\{\frac{\sin(n+1)\lambda x}{\sin\lambda x}\right\}^{l}\left\{\frac{\sin(p+1)\mu x}{\sin\mu x}\right\}^{m}..$$
$$= \frac{\pi c^{-hz}}{2}\left\{\frac{1-c^{-(n+1)2h\lambda}}{1-c^{-2h\lambda}}\right\}^{l}\left\{\frac{1-c^{-(p+1)2h\mu}}{1-c^{-2h\mu}}\right\}^{m}....(16)$$

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$$\int_0^{\frac{\pi}{2}} \frac{x\,dx \sin(z+nl\lambda+pm\mu..)x}{1-2r\cos 2x+r^2} \left\{\frac{\sin(n+1)\lambda x}{\sin\lambda x}\right\}^l \left\{\frac{\sin(p+1)\mu x}{\sin\mu x}\right\}^m ..$$

$$= -\frac{\pi}{4}\,\frac{r^{\frac{z}{2}}\log r}{1-r^2}\left\{\frac{1-r^{(n+1)\lambda}}{1-r^{\lambda}}\right\}^l \left\{\frac{1-r^{(p+1)\mu}}{1-r^{\mu}}\right\}^m \ldots\ldots (17)$$

$$\int_0^{\infty} \frac{x\,dx \sin(z+nl\lambda+pm\mu..)x}{h^2-x^2} \left\{\frac{\sin(n+1)\lambda x}{\sin\lambda x}\right\}^l \left\{\frac{\sin(p+1)\mu x}{\sin\mu x}\right\}^m ..$$

$$= -\frac{\pi}{2}\cos(z+nl\lambda+pm\mu..)h \left\{\frac{\sin(n+1)h\lambda}{\sin h\lambda}\right\}^l \left\{\frac{\sin(p+1)h\mu}{\sin h\mu}\right\}^m ..(18)$$

In all these integrals z must be positive and cannot be zero. Many other forms might have been given to $f(x)$; but in most cases it would not be practicable to represent the values of the integrals, except by means of symbols of operation. Indefinite integrals might also be found in this way; other symbols of operation might be employed; and several such symbols might be combined.

We may differentiate for l, m, *etc.*, and we may do this successively. I will put down some examples in concluding this paper.

$$\int_0^{\infty} \frac{dx\,(\cos\lambda x)^l \cos(z+l\lambda)x \log(\cos\lambda x)}{h^2+x^2}$$

$$= \frac{\pi c^{-hz}}{2h}\left(\frac{1+c^{-2h\lambda}}{2}\right)^l \log\left(\frac{c^{h\lambda}+c^{-h\lambda}}{2}\right) ..(19)$$

$$\int_0^{\infty} \frac{x\,dx\,(\cos\lambda x)^l \sin(z+l\lambda)x \log(\cos\lambda x)}{h^2+x^2}$$

$$= \frac{\pi}{2}\,c^{-hz}\left(\frac{1+c^{-2h\lambda}}{2}\right)\log\left(\frac{c^{h\lambda}+c^{-h\lambda}}{2}\right)..(20)$$

$$\int_0^{\infty} \frac{x\,dx\,(\cos\lambda x)^l \sin(z+l\lambda)\,x \log(\cos\lambda x)}{h^2-x^2}$$

$$= -\frac{\pi}{2}\left(\cos h\lambda\right)^l \cos\left(z+l\lambda\right) h \log\left(\cos h\lambda\right).. (21)$$

$$\int_0^{\frac{\pi}{2}} \frac{dx\,(\cos\lambda x)^l \cos(z+l\lambda)x \log(\cos\lambda z)}{1-2p\cos 2x+p^2}$$

$$= \frac{\pi}{2}\,\frac{p^{\frac{z}{2}}}{1-p^2}\left(\frac{1+p^{\lambda}}{2}\right)\log\left(\frac{p^{\frac{\lambda}{2}}+p^{-\frac{\lambda}{2}}}{2}\right)\ldots\ldots (22)$$

$$\int_0^{\frac{\pi}{2}} \frac{x\,dx\,(\cos\lambda x)^l \sin(z+l\lambda)x \log(\cos\lambda x)}{1-2p\cos 2x+p^2}$$

$$= -\frac{\pi}{4}\log p\,\frac{p^{\frac{z}{2}}}{1-p^2}\left(\frac{1+p^{\lambda}}{2}\right)\log\left(\frac{p^{\frac{\lambda}{2}}+p^{-\frac{\lambda}{2}}}{2}\right)..(23)$$

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ON SURFACES OF THE SECOND ORDER THAT HAVE THE SAME CURVE OF INTERSECTION.*

[*Mr. Fenwick.*]

1. *All surfaces of the second order which pass through the same eight points have in general the same curve of intersection.*

For let $u = 0$ and $u' = 0$,

be the equations of any two surfaces of the second order, passing through the same eight points; then λ being an arbitrary multiplier, every third surface passing through the same eight points has for its equation

$$u + \lambda u' = 0.$$

Now as this equation is satisfied by the co-ordinates of all the points which satisfy at the same time the equations $u = 0$, $u' = 0$, it follows, that this third surface contains all the points common to the other two surfaces, that is, it contains the curve of intersection of those surfaces.

2. *If any number of surfaces of the second order intersect in a plane curve, their second intersection (if they have one) is also a plane curve.*†

Let

$$u = ax^2 + by^2 + cx^2 + 2dxy + 2exz + 2fyz + 2gz + 2hy + 2kx + 1 = 0 \ldots (1)$$

$$u' = a'x^2 + b'y^2 + c'x^2 + 2d'xy + 2e'xz + 2f'yz + 2g'z + 2h'y + 2k'x + 1 = 0 \ldots (2)$$

be two surfaces that have the same curve of intersection. Then λ being an arbitrary multiplier,

$$u + \lambda u' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

will represent any third surface passing through the same points as (1) and (2).

If now we suppose these surfaces to intersect in a plane curve in the plane of xy, we must combine (3) with $z = 0$. This gives the equation

$$(b+\lambda b')y^2 + (c+\lambda c')x^2 + 2(d+\lambda d')xy + 2(h+\lambda h')y + 2(k+\lambda k')x + 1 + \lambda = 0,$$

or $$by^2 + cx^2 + 2dxy + 2hy + 2kx + 1 = 0,$$

and $$b'y^2 + c'x^2 + 2d'xy + 2h'y + 2k'x + 1 = 0.$$

Now as the surfaces intersect in the plane of xy, the preceding equations must be *identical*. Hence we have

$$b = b', \quad c = c', \quad d = d', \quad h = h', \quad k = k' \ldots\ldots\ldots\ldots (a)$$

Taking therefore (2) from (1), and affecting the result with the relations (a), we get for the line of intersection of the surfaces, the equation

$$(a - a')z^2 + 2(e - e')xz + 2(f - f')yz + 2(g - g')z = 0,$$

or $z = 0$, and $(a - a')z + 2(f - f')y + 2(e - e')x + 2(g - g') = 0$;

the latter of which shews that a second intersection is also a plane curve.

3. *To find the locus of the centres of the innumerable surfaces of the second order that have eight points common in space.*

* We shall confine ourselves in this paper to those surfaces of the second order that pass through the same eight points, as there may be other surfaces that have the *same curve of intersection*.

† This theorem for *two* surfaces is given by most of the French writers on curve surfaces (see also *Gregory's Solid Geometry*, p. 95), but I am not aware that any of the other properties of surfaces here investigated have been noticed.

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Denote the surfaces as in the last article.

Then the co-ordinates of the centre of (3) will be determined by the three equations (found by transformation, or by partial differentiation),

$$(e+\lambda e')z+(d+\lambda d')y+(c+\lambda c')x+(k+\lambda k')=0,$$
$$(f+\lambda f')z+(b+\lambda b')y+(d+\lambda d')x+(h+\lambda h')=0,$$
$$(a+\lambda a')z+(f+\lambda f')y+(e+\lambda e')x+(g+\lambda g')=0.$$

Eliminating λ from these, we have for the equations of the locus,

$$\left.\begin{aligned}&(az+fy+ex+g)(f'z+b'y+d'x+h')\\ =&(a'z+f'y+e'x+g')(fz+by+dx+h),\\ &(az+fy+ex+g)(e'z+d'y+c'x+k')\\ =&(a'z+f'y+e'x+g')(ez+dy+cx+k)\end{aligned}\right\}\ldots\ldots\ldots(4)$$

Now the former of these is satisfied by the following equations of two planes (λ being an arbitrary constant),

$$az+fy+ex+g=\lambda(a'z+f'y+e'x+g'),$$
$$f'z+b'y+d'x+h'=\frac{1}{\lambda}(fz+by+dx+h),$$

which evidently meet in a straight line that varies its position in consequence of the arbitrary constant λ. That surface is therefore generated by a straight line. It would be easy to shew also that it is *central*, and hence it is the *hyperboloid of one sheet*, as that is the only central surface of the second order that admits of a rectilinear generator.

In a similar way we might shew that the second of the equations (4) is the hyperboloid of one sheet. It follows, therefore, that *the centres under consideration lie on the intersection of two hyperboloids of one sheet.*

We shall deduce one or two properties from the general one, which may be interesting to the student.

Let the surfaces be *similar*, or

$$a=a',\quad b=b',\quad c=c',\quad d=d',\quad e=e',\quad f=f';$$

the two equations (4) then become *planes, which intersect in a line*, and as *all spheres are similar*, we have the following corollaries:

Cor. 1. *The centres of all similar surfaces of the second order that have the same curve of intersection, are in the same straight line.*

Cor. 2. *If any number of spheres intersect in the same circle, their centres will be in a straight line.*

4. *We shall next investigate the corresponding locus in plano.* It is given in the *Problémes et Développemens sur Diverses Parties des Mathématiques* of M. Reynaud and M. Duhamel (p. 348), but in a less general form than the following.

Let $v=ay^2+2bxy+cx^2+2dy+2ex+1=0$ .....(5)

$v'=a'y^2+2b'xy+c'x^2+2d'y+2e'x+1=0$ .....(6)

be two curves which pass through any four points in a plane, and λ an arbitrary multiplier, then

$$v+\lambda v'=0\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

is any third curve passing through the same points.

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Hence we have for the co-ordinates of the centre of (7),

$$(b+\lambda b')y+(c+\lambda c')x+(e+\lambda e')=0,$$
$$(a+\lambda a')y+(b+\lambda b')x+(d+\lambda d')=0.$$

Eliminating λ, there results for the locus of the centres, the equation

$$(ay+bx+d)(b'y+c'x+e')=(a'y+b'x+d')(by+cx+e),$$

which is a *curve of the second order*.

When $a=a'$, $c=c'$, $d=d'$, $e=e'$, the four points are in the axes of co-ordinates. The locus then becomes

$$ay^2-cx^2+dy-ex=0,$$

which is the *equation of a hyperbola passing through the origin*.

5. *To investigate the locus of the diameters conjugate to a given plane in respect of all surfaces of the second order that pass through eight points in space.*

The equations of a diameter to (3) article 2, conjugate to a plane $z=my+nx+p$, are

$$(f+\lambda f')z+(b+\lambda b')y+(d+\lambda d')x+(h+\lambda h')$$
$$+m\{(a+\lambda a')z+(f+\lambda f')y+(e+\lambda e')x+(g+\lambda g')\}=0,$$
$$(e+\lambda e')z+(d+\lambda d')y+(c+\lambda c')x+(k+\lambda k')$$
$$+n\{(a+\lambda a')z+(f+\lambda f')y+(e+\lambda e')x+(g+\lambda g')\}=0.$$

Eliminating λ, we have for the locus,

$$u_1u_2=u_3u_4,$$

where

$$u_1=(f+ma)z+(b+mf)y+(d+me)x+(h+mg),$$
$$u_2=(e'+na')z+(d'+nf')y+(c'+ne')x+(k'+ng'),$$
$$u_3=(f'+ma')z+(b'+mf')y+(d'+me')x+(h'+mg'),$$
$$u_4=(e+na)z+(d+nf)y+(c+ne)x+(k+ng).$$

Hence (art. 3) the locus is the *hyperboloid of one sheet*.

Scholium. It would seem from equation (52) page 287, Vol. I, of this work, that the theorems in articles 27, 28, are restricted to surfaces of the form $u_1u_2=\lambda u_3u_4$; u_1, u_2, *etc.*, being planes, and λ an arbitrary constant. This remark is due to our ingenious correspondent Mr. G. W. Hearn.

It is desirable, perhaps, that we should give in this place the different equations of surfaces of the second order which can be formed by the faces of a tetrahedron.

Let u_1, u_2, u_3, u_4, then, denote the faces ABD, BCD, CAD, BAC, of a tetrahedron ABCD, and let m, n, p, be arbitrary constants. Then the equation

$$u_1u_3+mu_1u_4+nu_2u_3+pu_2u_4=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (A)$$

whatever be the values of m, n, p, is satisfied by the equations $(u_1=0, u_2=0)$, $(u_3=0, u_4=0)$, which meet respectively in the lines AC, BD. The equation (A), therefore, expresses all surfaces of the second order which contain the edges AC, BD.

Again, the equation

$$u_1u_2+mu_1u_3+nu_1u_4+pu_2u_3=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (B)$$

represents all surfaces of the second order which contain the lines BD, AD, and the point C.

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For this equation is satisfied by the equations $(u_1=0, u_2=0)$, $(u_1=0, u_3=0)$, $(u_2=0, u_3=0, u_4=0)$, which meet, respectively, in the edges DB, AD, and the point C.

Also, the surface

$$u_1u_3 + mu_1u_4 + nu_2u_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (C)$$

being satisfied by $(u_1=0, u_2=0)$, $(u_1=0, u_3=0)$, $(u_3=0, u_4=0)$, contains the edges BD, AD, AC.

Lastly, the surface

$$u_1u_2 + mu_3u_4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (D)$$

contains the four straight lines expressed by the system

$$(u_1 = 0, u_3 = 0),\ (u_1 = 0, u_4 = 0),\ (u_2 = 0, u_3 = 0),\ (u_2 = 0, u_4 = 0).$$

These are obviously the lines AD, AB, CD, BC which form a quadrilateral in space.

ON THE TANGENTS OF ALGEBRAIC CURVES.

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If in an equation involving two variables, A_{pq} represent the coefficient of x^py^q, a plane curve of the n^{th} degree will be expressed by the equation

$$\Sigma(A_{pq}x^py^q) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which p and q are taken successively equal to any two whole positive numbers whose sum does not exceed n. The equation to a straight line is

$$y = mx + c \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Substituting in (1) $mx + c$ for y, the roots of the resulting equation are the abscissas of the points in which the straight line meets the curve. When two of the roots are equal they give a point at which the straight line *touches* the curve.

If both the curve and straight line pass through the origin of co-ordinates the constant terms in (1) and (2) vanish, and if at the same time the straight line is a tangent at the origin, then, when mx is substituted for y in (1), there must be two roots equal to zero, which cannot be the case unless the coefficient of the first power of x vanish.

Let (x',y') be a point in a curve the tangent at which is to be determined. Remove the origin to (x',y') by substituting in (1) $x'+x$ for x, and $y'+y$ for y. Then the value of m will be determined by substituting mx for y, and equating the co-efficient of x with zero.

By the binomial theorem and multiplication

$$(x'+x)^p\,(y'+y)^q = x'^p\,y'^q + px'^{p-1}\,y'^q x + qx'^p y'^{q-1} y + \ldots . (3)$$

Substituting the second member of (3) for $x^p\,y^q$ in (1), we find the terms involving the first powers of x and y to be

$$\Sigma(A_{pq}px'^{p-1}\,y'^q)x, \quad \Sigma(A_{pq}qx'^p y'^{q-1})y.$$

Put mx for y, and equate the sum of these with zero; then we find at once

$$m = -\frac{\Sigma(A_{pq}px'^{p-1}\,y'^q)}{\Sigma(A_{pq}qx'^p y'^{q-1})}$$

An expression indicating an operation identical with differentiation.

Having found m, the value of c in (2) is determined by the condition that the line passes through (x',y'); hence

$$c = y' - mx'.$$

The values of m and c thus found are such as render the line (2) a tangent to the curve (1) at the point x',y'.

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APPLICATION OF ALGEBRA TO THE MODERN GEOMETRY.

[*Mr. Robert Finlay, Professor of Mathematics and Natural Philosophy, Manchester New College.*]

(Continued from p. 6.)

XII.

To find the equation of a surface of the second degree passing through four of the intersections of four given planes.

If the given planes be denoted by

$$v_1 = a_1x + \beta_1y + \gamma_1z + 1 = 0 \ldots\ldots\ldots\ldots\ldots (1)$$
$$v_2 = a_2x + \beta_2y + \gamma_2z + 1 = 0 \ldots\ldots\ldots\ldots\ldots (2)$$
$$v_3 = a_3x + \beta_3y + \gamma_3z + 1 = 0 \ldots\ldots\ldots\ldots\ldots (3)$$
$$v_4 = a_4x + \beta_4y + \gamma_4z + 1 = 0 \ldots\ldots\ldots\ldots\ldots (4)$$

and if m be an arbitrary constant, the equation

$$mv_1v_3 = v_2v_4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

represents a surface of the second degree passing through the straight lines in which the planes (1) and (3) intersect (2) and (4). For equation (5) is evidently satisfied by the following systems of simultaneous equations,

$$\left.\begin{matrix} v_1 = 0 \\ v_2 = 0 \end{matrix}\right\} \quad \left.\begin{matrix} v_1 = 0 \\ v_4 = 0 \end{matrix}\right\} \quad \left.\begin{matrix} v_3 = 0 \\ v_2 = 0 \end{matrix}\right\} \quad \left.\begin{matrix} v_3 = 0 \\ v_4 = 0 \end{matrix}\right\}$$

which represent the four straight lines in question.

In like manner it may be demonstrated that each of the surfaces

$$m'v_1v_2 = v_3v_4 \text{ and } m''v_1v_4 = v_2v_3$$

passes through four of the intersections of the four given planes.

Since m, m', m'' are arbitrary, it is evident that *three systems of surfaces of the second degree may be described so that every curve in each system shall pass through four of the intersections of four given planes.*

XIII.

Let the planes (2), (3), and (4) be fixed, while (1) is variable; then if the surface (5) pass through two given points, the plane (1) will pass through a fixed point; but if the surface (5) pass through three given points, the plane (1) will pass through three given points which lie in the same straight line.

If we assume $v_2v_4 = tv_3$, equation (5) gives $mv_1 = t$, which may be written in the form

$$ax + by + cz + m = t \ldots\ldots\ldots\ldots\ldots\ldots (5')$$

Let $x'y'z'$, $x''y''z''$ be the co-ordinates of two given points, and let t' be the value of t when $x'y'z'$ are substituted for xyz; then the conditions in order that the surface (5′) may pass through the two given points are

$$ax' + by' + cz' + m = t',$$
$$ax'' + by'' + cz'' + m = t''.$$

From these we obtain by elimination

$$a(x't'' - x''t') + b(y't'' - y''t') + c(z't'' - z''t') + m(t'' - t') = 0,$$

from which it appears that the plane

$$ax + by + cz + m = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1')$$

passes through the fixed point

$$\frac{x't'' - x''t'}{t'' - t'}, \quad \frac{y't'' - y''t'}{t'' - t'}, \quad \frac{z't'' - z''t'}{t'' - t'} \ldots\ldots\ldots\ldots (a)$$

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When the plane (5′) also passes through a third point, $x''y''z''$, we have

$$ax''+by''+cz''+m=t'';$$

hence, in this case, the plane (1′) also passes through the two fixed points

$$\frac{x''t''-x''t'}{t''-t'}, \quad \frac{y''t''-y''t'}{t''-t'}, \quad \frac{z''t''-z''t'}{t''-t'} \quad \ldots\ldots\ldots\ldots (b)$$

$$\frac{x''t'-x't''}{t'-t''}, \quad \frac{y''t'-y't''}{t'-t''}, \quad \frac{z''t'-z't''}{t'-t''} \quad \ldots\ldots\ldots\ldots (c)$$

and the three points (a), (b), (c), are evidently in the same straight line.

XIV.

To find the equation to a surface of the second degree touching two given planes along the straight lines in which they are cut by a third plane.

When the planes (1) and (3) coincide, the four straight lines in which the surface (5) cuts the planes (2) and (4) evidently unite in two straight lines of contact. In this case equation (5) becomes $v_2v_4=mv_1^2$, or

$$v_2v_4=(ax+by+cz+e)^2 \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

where $a=a_1\sqrt{m}$, $b=\beta_1\sqrt{m}$, $c=\gamma_1\sqrt{m}$, $e=\sqrt{m}$; hence this equation denotes a surface of the second degree touching the planes (2) and (4) along the straight lines in which they are cut by the plane (1). The plane

$$ax+by+cz+e=0 \ldots\ldots\ldots\ldots\ldots\ldots (1')$$

which passes through the two lines of contact may be called the *plane of contact*, and, for the sake of brevity, the surface (6) may be said to have a *rectilinear plane contact* with the planes (2) and (4).

XV.

If each of two surfaces of the second degree have a rectilinear plane contact with two given planes, the surfaces cut each other on two planes, which pass through the intersection of the planes of contact, and form with those planes a system of four harmonic radial planes.

Let the two given planes be denoted by $v_1=0$, $v_2=0$, and the planes of contact by

$$ax+by+cz+e=0 \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

$$a'x+b'y+c'z+e'=0 \ldots\ldots\ldots\ldots\ldots\ldots (a')$$

then the equations of the surfaces will be (XIV)

$$v_1v_2=(ax+by+cz+e)^2 \ldots\ldots\ldots\ldots\ldots\ldots (b)$$

$$v_1v_2=(a'x+b'y+c'z+e')^2 \ldots\ldots\ldots\ldots\ldots\ldots (b')$$

Now by subtracting (b) from (b') we obtain

$$(a'x+b'y+c'z+e')^2-(ax+by+cz+e)^2=0,$$

which may be resolved into

$$(a'+a)x+(b'+b)y+(c'+c)z+e'+e=0 \ldots\ldots\ldots\ldots (c)$$

$$(a'-a)x+(b'-b)y+(c'-c)z+e'-e=0 \ldots\ldots\ldots\ldots\ldots (c')$$

consequently the surfaces (b) and (b') cut each other on the planes (c) and (c'), which evidently pass through the intersection of the planes of contact (a) and (a'), and form with them a system of four harmonic radial planes.

XVI.

If a surface of the second degree have a rectilinear plane contact with two given planes, and pass through two given points, the plane of contact will pass through a fixed point; but if it pass through three given points the plane of contact will pass through a fixed straight line.

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If the equations of the given planes be $v_1 = 0$, $v_2 = 0$, the equation of the surface will be (XIV)

$$v_1 v_2 = (ax + by + cz + e)^2,$$

and if $t = \sqrt{v_1 v_2}$, this becomes

$$ax + by + cz + e = t \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

The conditions in order that this surface may pass through the three given points $x'y'z'$, $x''y''z''$, $x'''y'''z'''$ being

$$ax' + by' + cz' = t',$$
$$ax'' + by'' + cz'' = t'',$$
$$ax''' + by''' + cz''' = t''',$$

it follows (XIII) that if the surface (a) pass through two of these points the plane of contact will pass through a given point, but if the surface (a) pass through the three given points the plane of contact will pass through three fixed points, which are in the same straight line.

XVII.

If a surface of the second degree meet a given surface of the second degree in two given plane sections, and if another surface of the second degree meet the given surface in two other plane sections, the intersection of these surfaces lies upon a surface of the second degree, which passes through four of the intersections of the four given planes.

Let $u = 0$ denote the given surface, $v = 0$, $v' = 0$, $v_1 = 0$, $v_2 = 0$ the planes of the given sections; the equations of the other two surfaces will be

$$u = mvv' \ldots\ldots (a),\ u = m_1 v_1 v_2 \ldots\ldots (a').$$

By eliminating u from these equations we get $mvv' = m_1 v_1 v_2$, consequently (XII) the intersection of the surfaces (a) and (a') lies on a surface of the second degree passing through the four straight lines in which the planes v and v' intersect the planes v_1 and v_2.

XVIII.

The following corollaries are perhaps worthy of being enumerated here.

Let $u = 0$ denote any given surface of the second degree, $v = 0$, $v' = 0$, $v'' = 0$, any three planes intersecting it in three curves (real or imaginary), a a fixed and m an arbitrary constant; then

1°. From the equations $u = mvv'$ and $u = avv''$ we deduce $v(mv' - av'') = 0$, which is equivalent to $v = 0$ and $mv' - av'' = 0$, and the last equation denotes a system of planes passing through the intersection of the planes $v' = 0$, and $v'' = 0$; hence we have the following theorem.

If a system of surfaces of the second degree pass through two given plane sections of a given surface of the second degree, and if another surface of the second degree pass through one of those sections and a third plane section, the latter surface will cut the surfaces of the system in a series of planes which pass through the same straight line.

2°. From the equations $u = mvv'$ and $u = av^2$ we obtain $v(mv' - av) = 0$;

Hence *if a system of surfaces of the second degree pass through two given plane sections of a given surface of the second degree, and if another surface of the second degree touch the given surface along one of those sections, the latter surface will cut the system of surfaces in*

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a series of planes, each of which passes through the intersection of the two given planes.

3° From the equations $u = mvv'$ and $u = av$ we get $v(mv' - a) = 0$; hence *if a system of surfaces of the second degree cut a given surface of the second degree in two given plane curves, any surface of the second degree similar to the given surface, which passes through one of those sections, will cut the system of surfaces in a series of planes, each of which is parallel to the other plane section.*

4°. From the equations $u = mv$ and $u = av^2$ we deduce $v(av - m) = 0$; hence *if a system of surfaces of the second degree cut a given surface of the second degree in a single plane curve, and if another surface of the second degree touch the given surface along that curve, the latter surface will cut the system of surfaces in a series of planes, each of which is parallel to the plane of contact.**

* The first of these is analogous to the following theorem. If any number of conic sections pass through four given points A, B, C, D, and if another conic section pass through A and B, but not through C and D, the straight lines joining the other two points in which it meets each of the former curves will intersect the straight line CD in the same point. *Poncelet, Propriétés Projectives, p.* 223.

(To be continued.)

ON THE GENERAL REDUCTION OF CERTAIN INTEGRALS.

II. General reduction of $\int \frac{x^m \log x\,dx}{(a+\beta x^2)^{n+\frac{1}{2}}}$, when m and n are positive or negative integers.

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(Continued from page 31, Vol. II.)

1. It will increase the interest of the function Ψ, to prove that this general integral is always reducible to $\overset{2}{L}$ and Ψ.

Let $\quad U_{m,n} = \frac{x^m \log x}{(a+\beta x^2)^{n-\frac{1}{2}}}; \quad V_{m,n} = \int \frac{x^m \log x\,dx}{(a+\beta x^2)^{n+}}.$

By differentiating $U_{m,n}$ and integrating back again, we get an equation of reduction which admits of three different forms :

$$U_{m,n} - \int \frac{x^{m-1}dx}{(a+\beta x^2)^{n-\frac{1}{2}}} = mV_{m-1,n-1} - (2n-1)\beta . V_{m+1,n} \quad \ldots\ldots\ldots\ldots (1)$$

$$= maV_{m-1,n} - (2n-m-1)\beta . V_{m+1,n} \quad \ldots\ldots\ldots (2)$$

$$= (2n-1)aV_{m-1,n} - (2n-m-1)\beta . V_{m-1,n-1} \quad \ldots (3)$$

When $m = 0$, the first gives $V_{1,n}$, but fails to reduce $V_{-1,n-1}$ to $V_{1,n}$: also the second then merges in the first. Yet by the third, $V_{-1,n-1}$ can be reduced to $V_{-1,n}$. When $m = 2n - 1$, the second and third formulæ coincide, and give $V_{2n-2,n}$, but fail to reduce $V_{2n-2,n-1}$ to $V_{2n-2,n}$.

There being no other cases of failure, and $V_{2n-2,n-1}$ being reducible by (1) to $V_{\pm 4,\pm 2}$, $V_{\pm 2,\pm 1}$, and finally to $V_{0,0}$, it will follow that $V_{m,n}$ is universally reducible to one of the three, $V_{1,0}$, $V_{0,0}$, $V_{-1,0}$, of which the first

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(being included under $V_{1,n}$) has been already found. The elementary integrals remaining are $V_{0,0}$ and $V_{-1,0}$, or $\int\frac{\log x\,dx}{\sqrt{a+\beta x^2}}$ and $\int\frac{\log x\,dx}{x\sqrt{a+\beta x^2}}$. In the latter, change x to $\frac{1}{y}$, and it becomes $\int\frac{\log y dy}{\sqrt{ay^2+\beta}}$; which merges in the former, a and β only being interchanged.

Nevertheless $V_{0,0}$ contains two different integrals according as the coefficient of x^2 is positive or negative.

2. First, let β be negative, therefore a is positive: put $\frac{\beta x^2}{a} = -z^2$, and dropping the known part of the integral which results, there is left as unknown

$$\int\frac{\log z dz}{\sqrt{1-z^2}} = \int\log\sin\omega d\omega, \text{ if } z=\sin\omega.$$

Thus we are again brought to the function which we have called $\Psi(\omega)$.

If we prefer to put $z=\cos\theta$, it gives $-\int\log\cos\theta d\theta$ as the resulting integral. When β on the contrary is positive, a may have either sign. Suppose a to be negative, and the form resulting is $\int\frac{\log z dz}{\sqrt{z^2-1}}$. Let $2z=\epsilon^{\theta}+\epsilon^{-\theta}$ (ϵ= base of Nap.logs.); and the integral is $\int\log\left(\frac{\epsilon^{\theta}+\epsilon^{-\theta}}{2}\right)d\theta$.

When a and β are *both* positive, let $\frac{\beta x^2}{a}=z^2$, and we have to integrate $\int\frac{\log z dz}{\sqrt{1+z^2}}$. Let $2z=\epsilon^{\theta}-\epsilon^{-\theta}$, and the integral is $\int\log\left(\frac{\epsilon^{\theta}-\epsilon^{-\theta}}{2}\right)d\theta$.

3. The close analogy between the two last and $\int\log\cos\theta d\theta$, $\int\log\sin\theta d\theta$, is manifest. Nevertheless it is not requisite to treat the exponential integrals at all; for both of them can be reduced to Spence's $\overset{2}{L}$.

First, let $Z=\int\frac{\log z dz}{\sqrt{z^2-1}}$ and $\sqrt{z^2-1}=zu$; then $z^{-2}=1-u^2$;

$\therefore -2\log z=\log(1-u^2)$, $4\log^2 z=\log^2(1-u^2)$;

$$\therefore 4\log z d\log z=\log(1-u^2).\frac{-2udu}{1-u^2}.$$

$$\text{Now } 4Z=\int\frac{4\log z.dz}{u.z}=\int\frac{4\log z.d\log z}{u}=-\int\log(1-u^2)\frac{2du}{1-u^2}$$

$$=-\int\{\log(1+u)+\log(1-u)\}.\left\{\frac{du}{1+u}+\frac{du}{1-u}\right\}$$

$$=-\int\log(1+u)\frac{du}{1+u}-\int\log(1-u)\frac{du}{1+u}-\int\log(1-u)\frac{du}{1-u}$$

$$-\int\log(1+u)\frac{du}{1-u}.$$

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Of these integrals, the first and third can be found immediately; and the second, integrated by parts, is (without the sign)

$$\log(1+u)\log(1-u)+\int\log(1+u)\frac{du}{1-u};\text{ hence}$$

$$4Z=-\tfrac{1}{2}\log^2(1+u)-\log(1+u)\log(1-u)+\tfrac{1}{2}\log^2(1-u)-2\int\log(1+u)\frac{du}{1-u}$$

$$=-\tfrac{1}{2}\{\log(1+u)+\log(1-u)\}^2+\log^2(1-u)-2\int\log(1+u)\frac{du}{1-u}.$$

Also, $-2\int\log(1+u)\frac{du}{1-u}=2\log 2.\log(1-u)+2\overset{2}{\mathrm{L}}\left(\frac{1+u}{2}\right)$, observing that $d\overset{2}{\mathrm{L}}x=\frac{\log x\,dx}{x-1}$.

Moreover, $\log^2(1-u)+2\log 2.\log(1-u)=\{\log(1-u)+\log 2\}^2-\text{const.}$
$=\log^2(2-2u)-\text{const.}$

Hence $4Z=-\tfrac{1}{2}\log^2(1-u^2)+\log^2(2-2u)+2\overset{2}{\mathrm{L}}\left(\frac{1+u}{2}\right)$,

or $Z=-\tfrac{1}{2}\log^2 z+\tfrac{1}{4}\log^2(2-2\sqrt{1-z^2})+\tfrac{1}{2}\overset{2}{\mathrm{L}}(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-z^2})$.

By a similar process, if $\sqrt{1+x^2}=vx$;

$$4\int\frac{\log z\,dz}{\sqrt{1+x^2}}=-\tfrac{1}{2}\log^2(v^2-1)+\log^2(2v-2)+2\overset{2}{\mathrm{L}}\left(\frac{v+1}{2}\right).$$

4. On the whole then it appears that $\int\frac{x^m\log x\,dx}{(\alpha+\beta x^2)^{n+\frac{1}{2}}}$ is always reducible either to Ψ or to $\overset{2}{\mathrm{L}}$. If α and β are both positive, the final integral is certain to be $\overset{2}{\mathrm{L}}$; but if one of the two is negative, it depends on the relations of m and n whether $\overset{2}{\mathrm{L}}$ or Ψ is to be our last result.

III. ON THE PROPERTIES AND CALCULATION OF $\Psi(x)=-\int_0\log\sin x.dx$.

1. Since $\log\sin x$ is negative, the negative sign is prefixed, in order to make Ψ positive. We suppose x to lie between 0 and π, else the logarithm is imaginary: such are the natural limits of Ψ.

Since $\sin(\pi-x)=\sin x$, $d\Psi(\pi-x)=-d\Psi x$: whence we get

$$\Psi x+\Psi(\pi-x)=\Psi x:\text{ and }2\Psi\left(\frac{\pi}{2}\right)=\Psi(\pi)\ldots\ldots\ldots\ldots\ldots(A)$$

This reduces the case of $x>\frac{\pi}{2}$ to that of $x<\frac{\pi}{2}$. A table of Ψ would not need to be continued beyond $x=\frac{\pi}{2}$.

2. The most obvious expansion of Ψ is obtained by making $\sin x=y$, and integrating by parts:

$$\therefore\Psi x=-x\log y+\int\frac{dy}{y}.\sin^{-1}y,\text{ and developing }\sin^{-1}y,$$

$$\Psi x=-x\log\sin x+\left\{\frac{y}{1^2}+\frac{1}{2}.\frac{y^3}{3^2}+\frac{1.3}{2.4}.\frac{y^5}{5^2}+etc.\right\}\ldots.(B)$$

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which may be used in a few isolated cases, as when

$$y = \frac{1}{2}, \text{ and } \therefore x = \frac{\pi}{6} : \text{ or indeed when } y = \frac{1}{\sqrt{2}}, \text{ and } x = \frac{\pi}{4}.$$

But it is very inferior in power to a series which will afterwards be given.

3. When x is between θ and $\frac{\pi}{2}$,

$$-\log(2\sin x) = \frac{\cos 2x}{1} + \frac{\cos 4x}{2} + \frac{\cos 6x}{3} + etc.;$$

whence we obtain $\left(\text{if } x < \frac{\pi}{2}\right)$

$$\Psi x = x\log 2 + \tfrac{1}{2}\left\{\frac{\sin 2x}{1^2} + \frac{\sin 4x}{2^2} + \frac{\sin 6x}{3^2} + etc.\right\} \ldots\ldots(C)$$

Indeed at the extreme limit, where $x = \frac{\pi}{2}$, this gives

$$\Psi\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\log 2; \text{ and therefore } \Psi\pi = \pi\log 2 \ldots\ldots(D)$$

but the eq. (C) can seldom converge with any rapidity.

4. It may be worth while next to expand Ψx in powers of $\cos x$.

Let $\cos x = z$, $\therefore \Psi x = \int \frac{\log\sqrt{1-z^2}}{\sqrt{1-z^2}}\,dz$. Let $T = \frac{-\log(1-t)}{\sqrt{1-t}}$; or

$$\frac{dT}{dt}.(1-t) - \frac{T}{2} = \frac{1}{\sqrt{1-t}} = 1 + \tfrac{1}{2}t + \frac{1.3}{2.4}t^2 + \frac{1.3.5}{2.4.6}t^3 + etc.$$

Assume $T = A_1 t + A_2 t^2 + A_3 t^3 + etc.$;

and, to determine the co-efficients, substitute in the differential equation.

This gives $A_1 = 1$; $2A_2 = 1\frac{1}{2}A_1 + \frac{1}{2}$; $3A_3 = 2\frac{1}{2}A_2 + \frac{1.3}{2.4}$, and generally

$$nA_n = (n - \tfrac{1}{2})\,A_{n-1} + \frac{1.3.\ldots.(2n-3)}{2.4.\ldots.(2n-2)};$$

Reducing A_n to A_{n-1}, A_{n-2}, A_{n-3}..... A_2, A_1, we easily exhibit A_n in general terms, *viz.*

$$\tfrac{1}{2}A_n = \frac{1.3.5.\ldots.(2n-1)}{2.4.6.\ldots.\ldots 2n}\left\{1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}\right\} \ldots\ldots(E)$$

For very large values of n, the outer fraction converges to $\frac{1}{\sqrt{\pi n}}$, and the series to $\{const. + \frac{1}{2}\log(2n-1)\}$: or A_n (when $n = \infty$) tends to $\frac{1}{\sqrt{\pi n}}\{2C + \log 2n\} = 0$. Yet $\frac{A_{n+1}}{A_n} = 1$, when $n = \infty$; hence the convergence is dependent entirely on the smallness of t.

Regarding A_1 A_2.... as known, we have, (putting $t = z^2$)

$$\Psi(x) = -\tfrac{1}{2}.\int\{A_1 z^2 + A_2 z^4 + A_3 z^6 + etc.\}\,dz$$

$$= \frac{\pi}{2}\log 2 - \tfrac{1}{2}\left\{A_1.\frac{z^3}{3} + A_2.\frac{z^5}{5} + A_3.\frac{z^7}{7} + etc.\right\} \ldots\ldots(F)$$

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This will determine Ψx when x is small, or x nearly $= \frac{\pi}{2}$.

5. Other series are obtained as follows, for reducing $\Psi\left(\frac{\pi}{2} - x\right)$ to Ψx, or conversely.

Let $\tan\omega = \frac{x \sin a}{1 - x\cos a}$. It was then shown in 1st section eqq.(C) (D) (E) that $\sin a.\phi(x,a) = \omega \log x - \frac{x}{1^2}\sin a - \frac{x^2}{2^2}\sin 2a - \frac{x^3}{3^2}$ s in$a$—*etc.*

$$= \Psi(\omega + a) - \Psi\omega - \Psi a.$$

Now, *first*, let $a = \frac{\pi}{2}$; therefore $x = \tan\omega$, and we get;

since $\Psi\left(\omega + \frac{\pi}{2}\right) = 2\Psi\left(\frac{\pi}{2}\right) - \Psi\left(\frac{\pi}{2} - \omega\right)$:

$$\Psi\frac{\pi}{2} - \Psi\left(\frac{\pi}{2} - \omega\right) - \Psi\omega = \omega \log\tan\omega - \frac{\tan\omega}{1^2} + \frac{\tan^3\omega}{3^2} - \frac{\tan^5\omega}{5^2} + etc.$$

$$\ldots\ldots\ldots(G)$$

Cor. Let $\pi = \frac{\pi}{4}$, and we obtain the same series for $\Psi\frac{\pi}{4}$ as (C) will give.

Next, let $\omega + a = \frac{\pi}{2}$, therefore $x = \cos a$; and we have

$$\Psi\frac{\pi}{2} - \Psi\left(\frac{\pi}{2} - a\right) - \Psi a = \left(\frac{\pi}{2} - a\right)\log\cos a$$

$$- \frac{\cos a}{1^2}\sin a - \frac{\cos^2 a}{2^2}\sin 2a - \frac{\cos^3 a}{3^3}\sin 3a - etc. \ldots (H)$$

(To be continued.)

MODERN GEOMETRY.

[*Mr. T. S. Davies, F. R. S.*]

(Continued from p. 256, vol. i.)

CHAPTER. III.

The Rule Surfaces of the Second Order.

XXV.

The first regular attempt to classify surfaces of the second order was made by Euler: but it is to Hachette that we owe our present complete view of the subject. Amongst ourselves these surfaces are scarcely known (except the simplest cases) otherwise than as the surfaces represented by the equation of the second degree; and I know of no English work in which these surfaces are rigorously discussed, or any genesis given of them when viewed as surfaces formed according to specified laws. The cases of the ellipsoïd, paraboloïd and hyperboloïd of two sheets, are indeed familar enough, under the name of conoids, when of revolution; and the extension to the cases where they are not of revolution is easy enough of conception, by considering all the ordinates in reference to one of the meridian planes to be altered in a given ratio. Yet of even these figures, it would be impossible to point out an English work which developes the general pro-

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perties with systematic precision, by analysis or geometry. This is the more remarkable, as such a development by either method is attended with no extraordinary difficulty.

There are, however, two other surfaces of the second order, to which we scarcely ever find any distinct allusion in our books; and yet they are surfaces which are not only necessary for completing the series of the second order, but otherwise possessed of properties so remarkable as to render them peculiarly attractive to the geometer. An elementary view of these will constitute the present chapter of modern geometry.

XXVI.

In order to discuss the surfaces of the second order geometrically, a geometrical genesis of them is essential.

1. Draw any tangent to the ellipse, parabola or hyperbola, and through this a plane perpendicular to that of the figure; let this plane move parallel to itself cutting the figure, and upon the chord so made in each position let a circle be described: then this variable circumference will respectively describe the *ellipsoïd*, *elliptic paraboloïd* and the *hyperboloïd of two sheets*.

This genesis is very analogous to that of an oblique cone; and in continuation of the same analogy, we may remark that the same surface may be described by the circle taken in one other special position, like that of the subcontrary section of the oblique cone. These circular sections have been aptly called by Chasles, the *cyclical sections*, and the planes parallel to them *cyclical planes*. They possess many curious and interesting properties.

2. If there be given in space three straight lines, no two of which are in the same plane, and upon these as directrices, another line constantly move, it will generate a surface of the second order, which (for a reason to be presently given) is called the *hyperboloïd of one sheet*.

3. If these three lines are parallel to the same plane, the surface generated by the fourth line is called the *hyperbolic paraboloïd*.

The last-named surface is sometimes defined by another mode of genesis: viz. by supposing the generating line to move upon two given lines, and be always parallel to a given plane. This, as will appear further on, is merely a consequence of the former hypothesis of the three directrices being parallel to another given plane.

XXVII.

To afford a general idea of the hyperboloid of one sheet, we will suppose the hyperbola to revolve about its imaginary axis (that is, one at right angles to the real axis of the curve, and passing through its centre) which is very easy of conception. This hyperbola will describe the hyperboloid of *revolution* of one sheet. It will require little effort of imagination to conceive that a line *may* rest wholly upon this surface in a position oblique to the axis. If the asymptotic angle in which the hyperbola is situated be very obtuse, the surface itself will very much resemble the ordinary shape of a dice-box. Under such an aspect it will be advisable for those who are not familiar with the general figure, to first view it; after which the idea can easily be extended to the general case.

There is another simple aspect under which the genesis of the hyper-

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boloïd of revolution of one sheet may be viewed. If there be given two straight lines which are not in the same plane, invariably connected with each other; and if one of these be made an axis of revolution, the other will generate this surface. In this way a model of the surface may be turned upon a common lathe, by an oblique position of the cutting edge with respect to the axis of motion.* When the one line meets the other, the surface becomes the right cone: and when they are parallel, the right cylinder.

XXVIII.

If three lines be given, no two of which are in the same plane, no point can be selected in one of them, through which a line cannot be drawn to touch the other two, and in no case can there be more than one such line.

For through this point and each line one plane can be drawn; and the intersection of these planes will be a line which cuts each of the lines through which the planes are drawn. Whence one such line can be always drawn.

Moreover, as no other planes than these can be drawn through that point and the two other lines, and these planes can have but one intersection, there can be only one such line drawn through each point of the third line.

It will, therefore, follow that the line in its motion upon the three directrices will generate a surface having no double or multiple point, since only one generatrix can ever pass through the same point.

It will also follow that the three directrices themselves are situated on the surface since they are touched by the generatrices in every position which they take.

It will still further follow that no two positions of the generating line, however closely contiguous, are in the same plane. For were that supposition admitted, the three lines upon which they rested would be in that plane, which is contrary to the original conditions. †

XXIX.

Let AB, CD, EF *be three directrices, no two of which are in the same plane; and let* AE, BF, GK, *be three positions of the generatrix: then we shall always have the relation amongst the segments of the twisted quadrilateral* ABFE,

$$AC.DF : CE.BD :: AG.KF : GB.EK.$$

Join AF, and produce it if necessary. Then the quadrilateral may be viewed as being composed of the triangles ABF, AEF, situated in two planes whose intersection is AF.

* This mode of generating the surface was invented by Sir Christopher Wren.

† If two of the directrices be parallel, the hyperboloïd is reduced to a plane, generated by a line moving along the third directrix parallel to the two former ones. For the intersection of the two supposed planes in the reasoning above, is always parallel to the lines through which those planes pass.

The same is also true when two of the directrices meet in a point; for then the locus is the plane which is drawn through that point and the third directrix.

We may add that the generatrix may take positions parallel to each of the directrices—one position parallel to each of them; and that it can take no more than one. For through any two of the directrices, planes may be drawn parallel to the third; and these planes will intersect in a generatrix which is parallel to that third directrix,—a well known proposition in geometry. Also as only one plane can be drawn through each of the two directrices parallel to the third, there can be but one pair of such planes, and therefore only one such generatrix.

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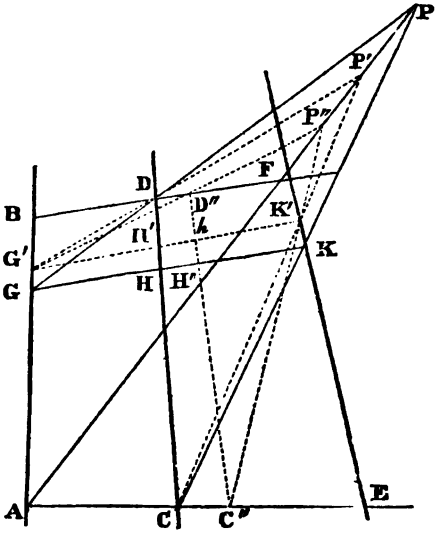
Now, since GK, CD, intersect in H, they are in one plane, and the lines GD, CK are also in one plane. Hence, the three lines GD, CK, AF are the intersections of these planes two and two; and they therefore meet in one point,—*viz.* that in which the planes themselves meet. Let this be P. Then the conclusion follows at once from Hutton, *prop.* xv. p. 228 of *vol.* ii. twelfth edition.



The first and second terms of this proportion are given when the directrices AB, CD, EF and two positions of the generatrix, AE, BF are given. These two can be always found by taking two points, as E, F in one of the directrices and drawing the planes EAB, ECD, which will intersect in ACE, and the planes FAB, FCD which will similarly intersect in ADF. The remaining generatrices may be constructed with great facility by the aid of the property just deduced.

For, take any point P in AF, the intersection of the planes ABF, AEF, and draw PDG, PKC to cut AB in G and EF in K respectively: then since these lines meet in P, they are in one plane. Wherefore CD, GK are also in one plane, and hence again the line GK which touches AB, EF also touches the third directrix CD in H. That is, GK is one position of the generatrix.

In like manner any other point P' in AF will give a new position G'H'K' of the directrix; and any number of them therefore may be constructed in a similar manner.

Scholium.—The general form of this proposition is often written

$$\frac{CA}{CE} : \frac{DB}{DF} :: \frac{GA}{GB} : \frac{KE}{KF}, \text{ or } \frac{KE}{KF} = m.\frac{GA}{GB};$$

a form which has advantages in some classes of inquiries, and which is much easier of recollection than that given in the first enunciation. The value of m is obviously deduced from the first two terms of the proportion.

XXX.

The hyperboloïd of one sheet admits of a second genesis,—viz, by taking any three generatrices AC, GK, BF *as directrices, in which case the three original directrices* AB, CD, EF *will become three positions of the new generatrix.*

It is, in the first place clear that the three lines AE, GK, BF being taken as directrices, no two of them being in the same plane (xxviii), the lines AB, CD, EF will be three positions of the generatrix of a surface of the *same species* as the former one, its genesis being precisely of the same kind. It remains to prove that by both modes, the *same surface* is produced. This will be effected if we can shew that every line of the second mode of genesis meets every line of the first mode: for then every point in

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the second mode coincides with a point in the first, and every line on the first surface is composed of points situated on the second.

Now the quadrilateral ABFE is the same, whichever pair of opposite sides AB, EF or AE, BF be considered to be directrices. Let then any fourth line of the surface G'K' be constructed by the first mode of genesis, and take any point P'' in AF; draw P''G', P''K' meeting BF in D'' and AE in C'': then D''C'' is a line of the second genesis, and meets GK in H'', it being constructed in reference to AE, GK, BF as directrices, in the same way that G'K' was in reference to AB, CD, BF. Whence by the previous property of the quadrilaterals (XXIX) we have

$$\begin{aligned} &CA.DF : CE.DB :: GA.KF : GB.KE, \\ &G'A.K'F : K'E.G'B :: CA.DF : CE.DB \\ &GA.KF : GB.KE :: C''A.D''F : C''E.D''B; \end{aligned}$$

and compounding these we get

$$G'A.K'F : K'E.G'B :: C''A.D''F : C''E.D''B.$$

It follows then from the converse of Hutton prop. XV. (which is too easily proved, ex absurdo, to need a formal specification here) that the lines G'D'', C''K' meet in the same point P'' of the diagonal AF. Whence they are in one plane, and the two lines G'K', C''D'' being also in that plane, they will meet in a point h.

Wherefore, since G'K' is *any* line of the first genesis and C''D'' *any* line of the second, and it has been shown that they will meet, the proposition is fully established.

XXXI.

The surface itself admitting of this double genesis, we see that through every point in it, two lines can be drawn which shall lie wholly upon the surface, one by each mode of genesis; and since no two lines of the same genesis can meet, there can be only two such lines through any point. Also, since every line of one genesis meets every line of the other, the surface itself may be imagined to be covered over with a species of net work of infinitely small quadrilaterals, the opposite sides of each being in different planes, however minute they may be.

The idea which we thus obtain of the hyperboloïd of one sheet does not readily assimilate with the idea of a surface of which the hyperboloïd of revolution of one sheet is a special case, or of a surface all whose plane sections are conic sections. On this account it seems necessary to continue the investigation to the extent of shewing that however tortuous and irregular the surface as resulting from this genesis may appear to the mind, it yet possesses regular and determinate features quite as simple in conception and expression as are possessed by the more familiar class of surfaces of the second order.

(To be continued.)

MATHEMATICAL NOTE.

To investigate the locus of the equation

$$(ay + bx + c)(a'y + b'x + c') = (dy + ex + f)(d'y + e'x + f');$$

the twelve coefficients a, a', etc., being independent of each other.

The equation of the locus is satisfied by the equations

$$\left.\begin{aligned} ay + bx + c &= \lambda\,(dy + ex + f) \\ a'y + b'x + c' &= \frac{1}{\lambda}(d'y + e'x + f') \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (1),$$

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λ being an arbitrary constant. Now as each of these is the equation of a line, the two combined represent a point; and as λ admits of an indefinite number of values, the locus in question is a curve traced out by a point. It is obviously (from the form of the equation) a curve of the second order.

Again, the co-ordinates of the centre will be determined by the following equations (found in the usual way),

$$(a'b + ab' - d'e - de')y + 2(bb' - ee')x + b'c + bc' - e'f - ef' = 0,$$
$$2(aa' - dd')y + (a'b + ab' - d'e - de')x + a'c + ac' - d'f - df' = 0.$$

The common denominator of the values of x, y, found from these equations, is

$$\left\{b'e'\left(\frac{a}{e'} - \frac{d}{b'}\right) + a'd'\left(\frac{b}{d'} - \frac{e}{a'}\right)\right\}^2 + 4e'b'a'd'\left(\frac{e}{b'} - \frac{b}{e'}\right)\left(\frac{a}{d'} - \frac{d}{a'}\right) \ldots (2)$$

Hence if we suppose a, a', e, e', *very large*, this expression will approximate to

$$\{e'(b' - d) + a'(b - d')\}^2 + 4aa'ee',$$

the latter member of which is evidently greater than the former. Now as a, a', e, e' may be positive or negative, the expression in question may be positive or negative. But (2) cannot $= 0$, unless we have the relations

$$\frac{a}{e'} = \frac{d}{b'}, \frac{b}{d'} = \frac{e}{a'}, \frac{e}{b'} = \frac{b}{e'}, \frac{a}{d'} = \frac{d}{a'};$$

$$\text{or,} \quad \frac{a}{d} = \frac{b}{e} = \frac{d'}{a'} = \frac{e'}{b'};$$

which are inadmissible, as the quantities a, a', *etc.*, are supposed to be independent of each other.

The curve is therefore *central*, and hence it is the *hyperbola* or *ellipse*.

There is a remarkable analogy between this form of the equation of the hyperbola or ellipse and that of the hyperboloïd of one sheet (see p. 81).

S. FENWICK.

NOTE ON THE EQUATION OF THE TANGENT TO THE PARABOLA.

The ordinary equation of the tangent to a parabola at a given point involves the co-ordinates of that point. It may, however, be put in another form involving only the angle which the tangent makes with the axis of x, and in this shape it possesses some advantages, and may be usefully employed in the domonstration of several theorems relating to that curve.

Let α denote the angle which a line cutting the parabola makes with the axis of x; then the equation of the line will be

$$y = x \tan \alpha + \beta \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and the equation of the parabola is

$$y^2 = 4mx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Treating these as simultaneous equations, we have

$$(x \tan \alpha + \beta)^2 = 4mx,$$

$$\text{or} \quad x^2 \tan^2 \alpha + 2(\beta \tan \alpha - 2m)x + \beta^2 = 0.$$

Now the roots of this equation are the values of x at the points of intersection of the line with the curve, and if the secant become a tangent, the two values of x must be equal: hence we must have the condition

$$4\beta^2 \tan^2 \alpha = 4(\beta \tan \alpha - 2m)^2.$$

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Extracting the root, gives

$$\pm \beta \tan a = \beta \tan a - 2m, \text{ or } 2m = \beta(\tan a \mp \tan a).$$

The value of m cannot be zero; therefore the sign — must be rejected, and using the sign + we get

$$\beta = m \cot a.$$

Hence the equation of the tangent may be put in the form

$$y = x \tan a + m \cot a \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where a is the angle which the tangent makes with the axis of x.

As an instance of the application of this equation, let

$$y = x \tan a + m \cot a \ldots\ldots\ldots\ldots\ldots\ldots (A)$$

be the equation of a tangent to a parabola; then the equation of a line drawn from the focus perpendicular to the tangent is

$$y = -x \cot a + m \cot a \ldots\ldots\ldots\ldots\ldots\ldots (B)$$

Consequently the lines (A) and (B) intersect in the point whose co-ordinates are $x = 0$, $y = m \cot a$; that is, in the tangent at the vertex.

R.

SOLUTIONS OF MATHEMATICAL EXERCISES.

XLIX. *James Cockle, M. A., Middle Temple.*

Shew that if $v = 3.2^m - 2$, then (see p. 114), $f^3(v)$ may be decomposed into the sum of m cubes; and make use of this result to obtain a similar theorem for $f^4(v)$.

[Solution. *By the Proposer.*]

Following the method pointed out at that page, we have

$$f^3(v) = h^3_1 + f^3\left(\frac{v-2}{2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

so that if $v = u_x$, and $\dfrac{v-2}{2} = u_{x+1}$,

$$u_{x+1} - \tfrac{1}{2} u_x = -1 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and the solution of this equation in finite differences is

$$u_x = (\tfrac{1}{2})^{x-1} \Sigma (-2^x) + C(\tfrac{1}{2})^{x-1} \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$= -2 + C(\tfrac{1}{2})^{x-1} \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Let $1 = u_{m+1} = -2 + C(\tfrac{1}{2})^m$, $\therefore C = 3.2^m \ldots\ldots\ldots\ldots (5)$

then $v = u_1 = 3.2^m - 2 \ldots\ldots\ldots\ldots\ldots\ldots (6)$

This might have been proved without the calculus of finite differences. The theorem for $f^4(v)$ will be derived from the same principles as the above.

LIV. *Mr. G. W. Hearn, R. M. College, Sandhurst.*

The greatest ellipsoid which can be inscribed in a given tetrahedron has its axes coincident with the *mechanical* principal axes (through the centre of gravity) of the tetrahedron.

[Solution. *By the Proposer.*]

Let E denote the volume of the ellipsoid, and T that of the tetrahedron; then these magnitudes will be connected by a certain equation

$$f\{E, T, h, k, l, etc.\} = 0,$$

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in which h, k, l, *etc.* are certain arbitrary quantities depending on the relative positions of the ellipsoid and tetrahedron, and which may be considered either as functions of the constants which enter into the equation of the ellipsoid, or as functions of the constants entering into the equations of the planes of the tetrahedron, according as we wish to determine the maximum ellipsoid which can be inscribed in a given tetrahedron, or the minimum tetrahedron which can be circumscribed about a given ellipsoid. In the former case we have T a constant and $d\mathrm{E} = 0$; in the latter, E a constant and $d\mathrm{T} = 0$, so that, in either case, we have to differentiate the equation

$$f\{\mathrm{E}, \mathrm{T}, h, k, l, \textit{etc.}\} = 0,$$

regarding $d\mathrm{E}$ and $d\mathrm{T}$ each $= 0$, and are therefore necessarily led to the same results.

Hence when the ellipsoid is a maximum with reference to the tetrahedron, the tetrahedron is a minimum with reference to the ellipsoid.

Now a minimum tetrahedron circumscribing any oval solid may be found thus:

Let one of the planes of the tetrahedron be subject to change its position, and refer it to the other fixed planes intersecting in the sides a, b, c; then we have for the equation of the moveable plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Now suppose x, y, z to be the co-ordinates of the point of contact; then since the volume of the tetrahedron is a minimum, abc is a minimum, and we have

$$\frac{x\,da}{a^2} + \frac{y\,db}{b^2} + \frac{z\,dc}{c^2} = 0,$$

$$\frac{da}{a} + \frac{db}{b} + \frac{dc}{c} = 0,$$

$$\therefore \frac{x}{a^2} = \frac{\lambda}{a}, \quad \frac{y}{b^2} = \frac{\lambda}{b}, \quad \frac{z}{c^2} = \frac{\lambda}{c},$$

λ being an arbitrary factor.

Wherefore,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 = 3\lambda \qquad \therefore \lambda = \frac{1}{3},$$

$$\therefore x = \frac{1}{3}a, \quad y = \frac{1}{3}b, \quad z = \frac{1}{3}c,$$

but these are the co-ordinates of the centre of gravity of the moveable face of the tetrahedron.

The same must hold as to all the faces. Hence the minimum tetrahedron which can circumscribe any oval solid is that which is touched by the solid in the respective centres of gravity of its four faces. Hence also the maximum ellipsoid which can be inscribed in a given tetrahedron touches each of the four faces in their centres of gravity.

From this it is easily found that the equation of the maximum ellipsoid referred to axes through the centre of gravity of the tetrahedron and parallel to a, b, c, will be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{xy}{ab} + \frac{yz}{bc} + \frac{zx}{ca} = \frac{1}{24}.$$

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The equations for finding the geometrical principal axes of this solid; that is, for finding $r^2 = x^2 + y^2 + z^2$, a maximum or minimum subject to the above condition, are

$$\left.\begin{aligned} \frac{2x}{a^2} + \frac{y}{ab} + \frac{z}{ac} &= \lambda\{x + y\cos\gamma + z\cos\beta\} \\ \frac{2y}{b^2} + \frac{x}{ab} + \frac{z}{bc} &= \lambda\{y + x\cos\gamma + z\cos\alpha\} \\ \frac{2z}{c^2} + \frac{y}{bc} + \frac{x}{ac} &= \lambda\{z + x\cos\beta + y\cos\alpha\} \end{aligned}\right\} \ldots\ldots\ldots(1)$$

λ being a multiplier. From these $\frac{y}{x}$, $\frac{z}{x}$ become known independent of λ, and thus determine the positions of the axes, α being the angle between b and c, *etc.* We have now to find equations for determining the mechanical principal axes of the solid.

Let there be an axis whose equations are $y' = mx'$, $z' = nx'$, and let

$P = 1 + m^2 + n^2 + 2m\cos\gamma + 2n\cos\beta + 2mn\cos\alpha$,

$Q = (1 + m\cos\gamma + n\cos\beta)x + (m + \cos\gamma + n\cos\alpha)y + (n + \cos\beta + m\cos\alpha)z$,

$R = x^2 + y^2 + z^2 + 2xy\cos\gamma + 2xz\cos\beta + 2yz\cos\alpha$;

then the square of the distance between the points x, y, z and x', y', z' will be

$$Px'^2 - 2Qx' + R,$$

and this distance will be at right angles to the assumed axis, when it is a minimum, and therefore when $x' = \frac{Q}{P}$.

Hence the square of the distance $= R - \frac{Q^2}{P}$, and therefore if A be a function of the angles α, β, γ, such that $Aabc = T$, we have for the moment of inertia of the pyramid about the axis $y = mx$, $z = nx$,

$$A\iiint R\,dx\,dy\,dz - A\frac{\iiint Q^2\,dx\,dy\,dz}{P}.$$

Of these the first not involving m and n may be dispensed with, and we will have to find the second a maximum or minimum with respect to m and n; that is, to investigate the equations

$$\frac{d}{dm}\cdot\left(\frac{\iiint Q^2\,dx\,dy\,dz}{P}\right) = 0,$$

$$\frac{d}{dn}\cdot\left(\frac{\iiint Q^2\,dx\,dy\,dz}{P}\right) = 0.$$

The values of the six integrals involved in these are

$$\iiint x^2\,dx\,dy\,dz = \frac{a^2.abc}{160}; \quad \iiint xy\,dx\,dy\,dz = -\frac{ab.abc}{480};$$

$$\iiint y^2\,dx\,dy\,dz = \frac{b^2.abc}{160}; \quad \iiint xz\,dx\,dy\,dz = -\frac{ac.abc}{480};$$

$$\iiint z^2\,dx\,dy\,dz = \frac{c^2.abc}{160}; \quad \iiint yz\,dx\,dy\,dz = -\frac{bc.abc}{480}.$$

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Substituting these we may advantageously replace m and n by $\frac{y}{x}$ and $\frac{z}{x}$ and multiply by x^2; then consider $\frac{\iiint Q^2}{P}$ as a function of three independent variables, introducing as usual an arbitrary factor Λ.

In this way I have deduced the following conditions,

$$\begin{aligned}
&(3a^2 - ab\cos\gamma - ac\cos\beta)(x + y\cos\beta + z\cos\gamma)\\
&+(3b^2\cos\gamma - ab - bc\cos\beta)(y + x\cos\gamma + z\cos\alpha)\\
&+(3c^2\cos\beta - ac - bc\cos\gamma)(z + x\cos\beta + y\cos\alpha)\\
&\qquad = \Lambda\{x + y\cos\gamma + z\cos\beta\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)\\
&(3a^2\cos\gamma - ab - ac\cos\alpha)(x + y\cos\beta + z\cos\gamma)\\
&+(3b^2 - ab\cos\gamma - bc\cos\alpha)(y + x\cos\gamma + z\cos\alpha)\\
&+(3c^2\cos\alpha - ac\cos\gamma - bc)(z + x\cos\beta + y\cos\alpha)\\
&\qquad = \Lambda\{y + x\cos\gamma + z\cos\alpha\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)\\
&(3a^2\cos\beta - ab\cos\alpha - ac)(x + y\cos\beta + z\cos\gamma)\\
&+(3b^2\cos\alpha - ab\cos\beta - bc)(y + x\cos\gamma + z\cos\alpha)\\
&+(3c^2 - ac\cos\beta - bc\cos\alpha)(z + x\cos\beta + y\cos\alpha)\\
&\qquad = \Lambda\{z + x\cos\beta + y\cos\alpha\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)
\end{aligned}$$

for determining $\frac{y}{x}$ and $\frac{z}{x}$ independent of Λ, and thence the position of the principal mechanical axes. Now I say that these equations will determine the values of $\frac{y}{x}$ and $\frac{z}{x}$ as the preceding equations (1) do. For we may multiply equations (1) so as to reproduce the same as the first member of equation (2), and we will find for the coefficient of x on the other side

$$\begin{aligned}
&\frac{2}{a^2}\left\{3a^2 - ab\cos\gamma - ac\cos\beta\right\}\\
&+\frac{1}{ab}\left\{3b^2\cos\gamma - ab - bc\cos\beta\right\}\\
&+\frac{1}{ac}\left\{3c^2\cos\beta - ac - bc\cos\gamma\right\}
\end{aligned}$$

which is equal to 4. The coefficient of y is similarly found to be $4\cos\gamma$, and that of z to be $4\cos\beta$. Hence the multipliers Λ and λ are connected by the condition $\Lambda\lambda = 4$. The same follows from the other combinations, and hence either set of equations must determine the same values of $\frac{y}{x}, \frac{z}{x}$.

LV. *Mr. G. W. Hearn, R. M. College, Sandhurst.*

The sum of the squares of the axes of the maximum ellipses inscribed on the faces of any tretrahedron is to the sum of the squares of the axes of its inscribed maximum ellipsoid as sixteen is to three.

[SOLUTION. *By the Proposer.*]

Eliminating $\frac{y}{x}, \frac{z}{x}$ from the equations (1) in last exercise, we find for determining λ the cubic equation

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$$\left(\frac{2}{a^2}-\lambda\right)\left(\frac{2}{b^2}-\lambda\right)\left(\frac{2}{c^2}-\lambda\right)-\left(\lambda\cos\alpha-\frac{1}{bc}\right)^2\left(\frac{2}{a^2}-\lambda\right)$$
$$-\left(\lambda\cos\beta-\frac{1}{ac}\right)^2\left(\frac{2}{b^2}-\lambda\right)-\left(\lambda\cos\gamma-\frac{1}{ab}\right)^2\left(\frac{2}{c^2}-\lambda\right)$$
$$=2\left(\lambda\cos\alpha-\frac{1}{bc}\right)\left(\lambda\cos\beta-\frac{1}{ac}\right)\left(\lambda\cos\gamma-\frac{1}{ab}\right).$$

From which we easily find the sum of the reciprocals of the roots $=$
$$\tfrac{2}{3}(a^2+b^2+c^2)-\tfrac{1}{2}(ab\cos\gamma+ac\cos\beta+bc\cos\alpha)$$

Let d, e, f be the sides of the tretrahedron respectively opposite a, b, c.

Then $bc\cos\alpha=\frac{1}{2}\{b^2+c^2-d^2\}$, $ab\cos\gamma=\frac{1}{2}\{a^2+b^2-f^2\}$;
$$ac\cos\beta=\tfrac{1}{2}\{a^2+c^2-e^2\},$$
which will reduce the above to
$$\tfrac{1}{4}\{a^2+b^2+c^2+d^2+e^2+f^2\}.$$

But multiplying equations (1) by x, y, z, and adding, taking into account the equation to the ellipsoid, we have
$$\lambda r^2=\frac{1}{12};\quad \therefore\ r^2=\frac{1}{12\lambda}.$$

Hence $\Sigma r^2=\frac{1}{48}\Sigma a^2$; that is, the sum of the squares of the semi-axes of the ellipsoid $=\frac{1}{48}$ of the sum of the squares of the sides of the tetrahedron.

Now by a very easy process analogous to that developed in the last question, we have for the equation to a maximum ellipse inscribed in a triangle
$$\frac{x^2}{a^2}+\frac{xy}{ab}+\frac{y^2}{b^2}=\frac{1}{12},$$
a and b being two of its sides parallel to the axes of reference which are supposed to pass through the centre of gravity of the triangle.

The sum of the squares of the axes of this we find to be $\frac{1}{18}(a^2+b^2+c^2)$. Hence the sum of the squares of all the axes of the ellipses is $\frac{1}{9}(a^2+b^2+c^2+d^2+e^2+f^2)$, since each side is common to two faces;

$$\therefore\ \frac{\text{sum of the squares of the axes of the ellipses}}{\text{sum of the squares of the axes of the ellipsoid}}=\frac{48}{9}=\frac{16}{3}.$$

Note—This ratio was inadvertently stated to be 32 to 9.

LVI. *Mr. Rutherford.*

The upper extremity A of a rod AB, originally in a vertical position, is projected with a given velocity along a smooth groove inclined to the horizon at a given angle: determine the motion of the rod, and the time when the lower extremity B passes the horizontal line through the point of projection.

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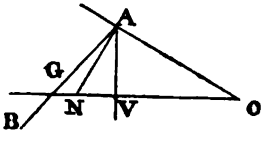
[SOLUTION. *Mr. G. W. Hearn, R.M. College, Sandhurst.*]

Let O be the point of projection; the horizontal line ON the axis of x, OA the groove; M the mass of the rod, a its length, k its radius of gyration about A; xy the co-ordinates of any point in the rod, and let AV, AN be respectively perpendicular to ON, OA. Let OA $= u$, VAB $= \theta$, NAB $= \Psi$, AOV $=$ VAN $= \alpha$, and $r =$ the distance of the point xy from A. Then $x = u\cos\alpha + r\sin\theta$, and $y = u\sin\alpha - r\cos\theta$. The dynamical condition is



$$\Sigma m\left\{\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + g\delta y\right\} = 0,$$

which, since δu and $\delta\theta$ are independent, resolves into

$$\Sigma m\left\{\cos\alpha\frac{d^2x}{dt^2} + \sin\alpha\frac{d^2y}{dt^2} + g\sin\alpha\right\} = 0,$$

and, $$\Sigma m\left\{r\cos\theta\frac{d^2x}{dt^2} + r\sin\theta\frac{d^2y}{dt^2} + gr\sin\theta\right\} = 0,$$

or substituting and reducing in the usual way, n^2 being put for $\frac{k^2}{a^2}$, we get for the two equations of motion,

$$\frac{d^2u}{dt^2} + a\frac{d^2\sin\Psi}{dt^2} + g\sin\alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and, $$\frac{d^2u}{dt^2}\cos\Psi + an^2\frac{d^2\Psi}{dt^2} + g\sin(\Psi + \alpha) = 0 \quad \ldots\ldots\ldots\ldots (2)$$

Eliminating $\frac{d^2u}{dt^2}$ from (1) and (2) we get for finding Ψ in terms of t, the equation

$$an^2\frac{d^2\Psi}{dt^2} - a\cos\Psi\frac{d^2\sin\Psi}{dt^2} + g\cos\alpha\sin\Psi = 0.$$

Multiply this by $2\frac{d\Psi}{dt}$, and integrate, then

$$a\left(n^2 - \cos^2\Psi\right)\frac{d\Psi^2}{dt^2} = 2g\cos\alpha\cos\Psi + C \quad \ldots\ldots\ldots (3)$$

C being a constant depending on the initial conditions.

To determine C, let V be the given velocity, P the impulsive force in direction OA necessary to produce the initial velocity V of the point A. Then we have for initial values

$$\Sigma m\left\{\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y\right\} - \mathrm{P}\cos\alpha.\Delta x - \mathrm{P}\sin\alpha\,\Delta y = 0,$$

where Δx, Δy are the δx, δy on the groove, so that $\Delta x = \delta u\cos\alpha$, and $\Delta y = \delta u\sin\alpha$;

$$\therefore \quad \Sigma m\left\{\frac{dx}{dt}\cos\alpha + \frac{dy}{dt}\sin\alpha\right\} = \mathrm{P},$$

$$\text{or} \quad \frac{du}{dt} + a\cos\Psi\frac{d\Psi}{dt} = \frac{\mathrm{P}}{\mathrm{M}}.$$

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But the initial value of Ψ is $-a$; therefore the last equation is

$$\frac{du}{dt} + a\cos a\,\frac{d\Psi}{dt} = \frac{P}{M}.$$

Also, $$\Sigma m\left\{\frac{dx}{dt}\,r\cos\theta + \frac{dy}{dt}\,r\sin\theta\right\} = 0,$$

or $$\cos a\,\frac{du}{dt} + n^2 a\,\frac{d\Psi}{dt} = 0.$$

Eliminating $\frac{d\Psi}{dt}$ we get

$$\frac{du}{dt} = \frac{P}{M}\cdot\frac{n^2}{n^2-\cos^2 a} = V,$$

$$\therefore \quad P = MV\left(1 - \frac{\cos^2 a}{n^2}\right) = \text{the impulsive force.}$$

Also, $\frac{d\Psi}{dt} = -\frac{V}{a}\cdot\frac{\cos a}{n^2}$, the initial angular velocity. Putting, therefore, in (3) $\Psi = -a$, and $\frac{d\Psi}{dt} = -\frac{V}{a}\cdot\frac{\cos a}{n^2}$, we obtain

$$C = \frac{V^2}{a}\cdot\frac{\cos^2 a}{n^4}(n^2-\cos^2 a) - 2g\cos^2 a.$$

Integrating equation (1) and correcting by the initial values, we have

$$u + a\sin\Psi = V\left(1 - \frac{\cos^2 a}{n^2}\right)t - \tfrac{1}{2}g\sin a.t^2 - a\sin a \ldots\ldots\ldots\ldots\ldots (4)$$

Now when a point of the rod at a distance b from A is in the horizontal line OV, we have

$$u = \frac{b\cos(\Psi + a)}{\sin a},$$

so that eliminating u from (4) we have another equation between Ψ and t, and then by integrating (3) we have the general value of Ψ in terms of t; whence on eliminating Ψ from (4) an equation will arise for determining t.

The integration of (3) however is in general impracticable in finite form, but there is one case in which it may be done, *viz.*, when $n^2 = 1$, that is, when the rod is considered without weight and is loaded with a heavy particle at its extremity B. The integral of (3) in this case is

$$\cos\Psi = \frac{g\cos a}{2a}t^2 - \frac{V}{2a}\sin 2a.t + \cos a;$$

and since $b = a$, equation (4) gives

$$a\cos\Psi\cot a = V\sin^2 a.t - \tfrac{1}{2}g\sin a.t^2 - a\sin a.$$

Eliminating $\cos\Psi$, we have

$$t^2 - \frac{2V\sin a}{g}.\,t + \frac{2a}{g} = 0;$$

$$\therefore\ t = \frac{V\sin a \pm \sqrt{V^2\sin^2 a - 2ag}}{g}.$$

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Or if h be the height due to the given velocity vertically resolved, and therefore $V^2 \sin^2 a = 2gh$, then

$$t = \sqrt{\frac{2}{g}} \cdot \{\sqrt{h} \pm \sqrt{(h-a)}\}.$$

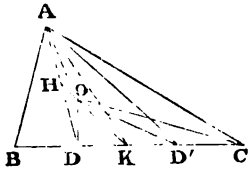
Cor. When h is greater than a, the particle at B twice attains the horizontal line; when $h = a$, it attains it once; and when h is less than a, never.

LVII. *Mr. J. W. Elliott, Greatham.*

If a straight line be drawn from the vertex of any plane triangle to the point of contact of either the inscribed or escribed circle; then the middle points of that line and the base, and the centre of the circle, range in the same straight line.

[First Solution. *The Rev. John Leech, Curate of Moy, Tyrone.*]

Let ABC be a plane triangle, K the middle of the base, O the centre of the inscribed circle, OD the radius perpendicular to BC. Draw AO, BO, CO, AD, AK, OK, and let AD′ be drawn parallel to OK; draw also OD′, and produce KO to meet CD in H. Then because AD′ is parallel to OK, the triangles AOD′ and AD′K are equal; and if to each of these equals the triangle ACD′ be added, the quadrilateral ACD′O will be equal to the triangle CAK, which is one-half of the triangle ABC. Again, if perpendiculars be drawn from O to the sides AC and AB, it will be evident that the triangles AOC and BOD are together equal to one-half of the triangle ABC; therefore the triangles AOC and BOD are together equal to the quadrilateral ACD′O. From each of these equals take the triangle AOC, and the remaining triangles COD′ and BOD are equal; therefore the base CD′ is equal to the base BD, and consequently KD′ is equal to KD. Lastly, because KOH is parallel to AD′, and DD′ is bisected in K; therefore AD is also bisected in H. In a similar manner may the property be established for any of the escribed circles.



[Second Solution. *Mr. R. Harley, Northwich, Cheshire.*]

Let O, O_1 be the respective centres of the inscribed and escribed circles; D, D′ their points of contact with the base BC of the triangle; DP, D′P_1 their diameters drawn perpendicular to BC; and K, H, H_1 the middle points of the lines BC, AD, AD′ respectively.

Then (*Modern Geometry, Student*, prop. 45, cor. 1, and prop. 58), KD = KD′, and the points A, P, D′ are in one straight line. Also the points A, D, P_1 are in one straight line; therefore the lines KH, KH_1 are respectively parallel to AD′, AD; consequently KH and H_1K produced bisect the diameters DP, D′P_1; that is, they pass through the centres O, O_1 of the inscribed and escribed circles.

[Third Solution. *Mr. John Laws, Newcastle-on-Tyne.*]

The base of the triangle being the axis of x, and its middle point the origin of rectangular co-ordinates; let x_1y_1, x_2y_2 be the co-ordinates of the centres

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of the inscribed and escribed circles; x_2y_2, x_4y_4 the co-ordinates of the middle points of the straight lines from the vertex to the points of contact of the circles with the base respectively. Also denote the three sides of the triangle by a, b, c, (a being the base) and let p = the perpendicular from the vertex to the base, and $2s = a + b + c$; then we have

$$x_1 = \frac{b-c}{2}, \quad x_2 = -\frac{b-c}{2}, \quad x_3 = \frac{s(b-c)}{2a}, \quad x_4 = \frac{(s-a)(b-c)}{2a},$$

$$y_1 = \frac{ap}{2s}, \quad y_2 = -\frac{ap}{2(s-a)}, \quad y_3 = \frac{p}{2}, \quad y_4 = \frac{p}{2}.$$

Hence the equations of the lines through x_1y_1, x_3y_3, and x_2y_2, x_4y_4 respectively, are

$$s(b-c)y = apx,$$
$$(s-a)(b-c)y = apx,$$

both of which pass through the origin of co-ordinates.

Correct and elegant geometrical solutions were given by Messrs. Samuel Bills; J. W. Elliott, the proposer; G. W. Hearn; W. H. Levy; Nicholas Smyth, and Thos. Weddle. Messrs. Henry Blythe; G. W. Hearn; and J. Wright sent good analytical solutions.

LVIII. *By F.*

Prove, without the aid of goniometry, the following relation which we may name the *Reciprocal Theorem:*

$$x^n + \frac{1}{x^n} = y^n - ny^{n-2} + \frac{n(n-3)}{1\cdot2}y^{n-4} - \frac{n(n-4)(n-5)}{1\cdot2\cdot3}y^{n-6} + \ldots.$$

where $y = x + \frac{1}{x}$.

[First Solution. *Mr. G. W. Hearn, R. M. College.*]

When n is a whole number, the reciprocal theorem may be established in the following manner.

Let $u_n = x^n + \frac{1}{x^n}$; then since $y = x + \frac{1}{x}$,

$$yu_n = x^{n+1} + \frac{1}{x^{n+1}} + x^{n-1} + \frac{1}{x^{n-1}},$$

or, $yu_n = u_{n+1} + u_{n-1}$; $\therefore \quad u_{n+1} = yu_n - u_{n-1}$.

By means of this equation it may be easily shewn that if the theorem be true for two consecutive values of n, it is true for the next superior value. The truth of it is easily recognized when $n = 1$ and $n = 2$, and therefore it will be true for $n = 3$. Therefore for $n = 4$, and so on. This is the proof by successive inductions.

But this method (and I think every other) is inferior in generality to the following.

Suppose $x = y - v.x^{-1}$.

Expand x^n in powers of v by Lagrange's theorem, and then put $v = 1$; we thus have

$$x^n = y^n - \left\{y^{-1}\frac{d(y^n)}{dy}\right\} + \frac{1}{1.2}\frac{d}{dy}\left\{y^{-2}\frac{d(y^n)}{dy}\right\}$$
$$- \frac{1}{1.2.3}\frac{d^2}{dy^2}\left\{y^{-3}\frac{d(y^n)}{dy}\right\} \ldots\ldots\ldots$$

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$$= y^n - ny^{n-2} + \frac{n.(n-3)}{1.2} y^{n-4} - \frac{n.(n-4).(n-5)}{1.2.3} y^{n-6} \ldots.$$

Now an apparent inconsistency presents itself. The right hand member is seemingly the same as the series given to be proved equal to $x^n + \frac{1}{x^n}$ and cannot at the same time be equal to x^n. But let us now develope x^{-n}. By the same Lagrange's theorem we have

$$x^{-n} = y^{-n} + ny^{-(n+2)} + \frac{n.(n+3)}{1.2} y^{-(n+4)} + \frac{n.(n+4)(n+5)}{1.2.3} y^{-(n+6)} \ldots.$$

Hence $x^n + \frac{1}{x^n}$ must be equal to the sum of those series *whatever be the value of n*, since this mode of proof is independent of integral values.

Now when n is a whole number it is easy to shew that the coefficient of y^{n-2r}, in the first series, is equal and of contrary sign to that of y^{-n-2s} in the second series, provided that

$$n - 2r = -n - 2s,$$

or,
$$r - s = n.$$

Hence when n is an integer, all the terms of the first series involving negative powers of y will destroy the corresponding powers in the second series. But since r and s are integers the condition $r - s = n$ cannot be satisfied when n is fractional, so that it will then require both series to express the true expansion of $x^n + \frac{1}{x^n}$.

Hence we ought to have

$$\cos n\theta = 2^{n-1} \left\{ \cos^n\theta - n\cos^{n-2}\theta + \frac{n.(n-3)}{1.2} \cos^{n-4}\theta \ldots. \right\}$$

when n is an integer the series terminating with $\cos\theta$ or a constant, so as to exclude all negative powers of $\cos\theta$.

But when n is a fraction as $\frac{p}{q}$,

$$\cos\frac{p}{q}\theta = 2^{\frac{p}{q}-1} \left\{ \cos\frac{2\lambda p\pi}{q} + \sqrt{-1}\sin\frac{2\lambda p\pi}{q} \right\}$$
$$\times \left\{ \cos^{\frac{p}{q}}\theta - \frac{p}{q}\cos\frac{p-2q}{q}\theta + \frac{p.(p-3q)}{1.2q^2}\cos\frac{p-4q}{q}\theta + etc. \right\}$$
$$+ 2^{-\left(\frac{p}{q}+1\right)} \left\{ \cos\frac{2\lambda p\pi}{q} + \sqrt{-1}\sin\frac{2\lambda p\pi}{q} \right\}$$
$$\times \left\{ \sec^{\frac{p}{q}}\theta + \frac{p}{q}\sec\frac{p+2q}{q}\theta + \frac{p.(p+3p)}{1.2.q^2}\sec\frac{p+4q}{q}\theta + \right\}$$

both series infinitely continued, and λ having the q values 0, 1, 2, 3..—.. $q-1$.

See Dr. Peacock's paper for Smith's Prize, Camb. Cal. for 1837.

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[SECOND SOLUTION. *Mr. Thomas Weddle, Newcastle; and similarly by Mr. S. Bills, Hawton.*]

Since $x + \frac{1}{x} = y; \quad x = \frac{1}{2}\{y \pm \sqrt{y^2-4}\};$

$$\therefore \quad x^n + \frac{1}{x^n} = \frac{1}{2^n}\{y+\sqrt{y^2-4}\}^n + \frac{1}{2^n}\{y-\sqrt{y^2-4}\}^n \ldots (1)$$

Now it is not difficult to see that the left hand member of (1) may be expanded in the form

$$x^n + \frac{1}{x^n} = y^n - Ay^{n-2} + By^{n-4} - \ldots\ldots\ldots$$

and that when n is even, the last term will be ± 2. Hence if we assume

$$x^n + \frac{1}{x^n} = y^n - \phi_1(n).\, y^{n-2} + \phi_2(n).\, y^{n-4} - \phi_3(n).\, y^{n-6} + \ldots (2)$$

we shall have $\phi_n(2n) = 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

Multiply the left hand member of (2) by $x + \frac{1}{x}$ and the right by its equal y,

$$\therefore x^{n+1} + \frac{1}{x^{n+1}} + x^{n-1} + \frac{1}{x^{n-1}}$$

$$= y^{n+1} - \phi_1(n).\, y^{n-1} + \phi_2(n).y^{n-3} - \phi_3(n).\, y^{n-5} + \ldots.$$

that is, by (2)

$$y^{n+1} - \{\phi_1(n+1) - 1\}.y^{n-1} + \{\phi_2(n+1) - \phi_1(n-1)\}y^{n-3} - \ldots.$$

$$= y^{n+1} - \phi_1(n).y^{n-1} + \phi_2(n).y^{n-3} - \phi_3(n).y^{n-5} + \ldots\ldots\ldots (4)$$

Equating the coefficients of (4), we readily get

$$\Delta\phi_1(n) = 1,$$
$$\Delta\phi_2(n) = \phi_1(n-1),$$
$$\Delta\phi_3(n) = \phi_2(n-1),$$
$$\textit{etc.} \qquad \textit{etc.}$$

Hence integrating and determining the constants so that (3), $\phi_1(2) = \phi_2(4) = \phi_3(6) = \ldots = 2$, we have

$$\phi_1(n) = n$$

$$\therefore \quad \Delta\phi_2(n) = n - 1 = (n-3) + 2$$

$$\therefore \quad \phi_2(n) = \frac{(n-3)(n-4)}{1.2} + 2(n-3) = \frac{n(n-3)}{1.2}$$

$$\therefore \quad \Delta\phi_3(n) = \frac{(n-4)(n-5)}{1.2} + 2(n-4)$$

$$\therefore \quad \phi_3(n) = \frac{(n-4)(n-5)(n-6)}{1.2.3} + \frac{2(n-4)(n-5)}{1.2} = \frac{n(n-4)(n-5)}{1.2.3}$$

$$\therefore \quad \Delta\phi_4(n) = \frac{(n-5)(n-6)(n-7)}{1.2.3} + \frac{2(n-5)(n-6)}{1.2}$$

$$\therefore \quad \phi_4(n) = \frac{(n-5)(n-6)(n-7)(n-8)}{1.2.3.4} + \frac{2(n-5)(n-6)(n-7)}{1.2.3}$$

$$= \frac{n(n-5)(n-6)(n-7)}{1.2.3.4}$$

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Proceeding in this way, we evidently have generally

$$\phi_m(n) = \frac{\{n-(m+1)\}\{n-(m+2)\}\ldots\ldots(n-2m)}{1.2.3\ldots\ldots m} + \frac{2\{n-(m+1)\}\{n-(m+2)\}\ldots\ldots\{n-(2m-1)\}}{1.2.3\ldots\ldots(m-1)} = \frac{n\{n-(m+1)\}\{n-(m+2)\}\ldots\ldots\{n-(2m-1)\}}{1.2.3..m}.$$

Hence we get

$$x^n + \frac{1}{x^n} = y^n - n.y^{n-2} + \frac{n(n-3)}{1.2}y^{n-4} - \ldots\ldots \pm \frac{n\{n-(m+1)\}\{n-(m+2)\}..\{n-(2m-1)\}}{1.2.3\ldots\ldots m}y^{n-2m}.$$

[THIRD SOLUTION. *Mr. Henry Blythe, of Mr. Jeffery's Academy, Broom Hall, Shooter's Hill; and similarly by Mr. R. Harley, Northwich, Cheshire.*]

It is easily seen that the developement is of the form

$$x^n + x^{-n} = y^n + A_1y^{n-1} + A_2y^{n-2} + A_3y^{n-3} + \ldots\ldots\ldots$$

Expanding the right hand member of this, we obtain

$$x^n + x^{-n} =$$

$$\begin{array}{l|l|l|

Taking the Naperian logarithms on each side,

$$\log_e(1-\beta x)+\log_e\left(1-\frac{\beta}{x}\right)=\log_e\{1-\beta(y-\beta)\}.$$

Changing the signs of both members, and expanding by the formula

$$\log_e(1-z)=-(z+\tfrac{1}{2}z^2+\tfrac{1}{3}z^3+\ldots.),$$

there results the equation

$$\left(x+\frac{1}{x}\right)\beta+\tfrac{1}{2}\left(x^2+\frac{1}{x^2}\right)\beta^2+\tfrac{1}{3}\left(x^3+\frac{1}{x^3}\right)\beta^3+\ldots.\frac{1}{n}\left(x^n+\frac{1}{x^n}\right)\beta^n..$$

$$=\beta(y-\beta)+\tfrac{1}{2}\beta^2(y-\beta)^2+\ldots.+\frac{1}{n-3}\beta^{n-3}(y-\beta)^{n-3}$$

$$+\frac{1}{n-2}\beta^{n-2}(y-\beta)^{n-2}+\frac{1}{n-1}\beta^{n-1}(y-\beta)^{n-1}+\frac{1}{n}\beta^n(y-\beta)^n+..$$

Now since by hypothesis β may have any value whatever, the coefficient of β^n on one side of the above equation must be equal to the coefficient of β^n on the other. That on the first side is

$$\frac{1}{n}\left(x^n+\frac{1}{x^n}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The terms involving β^n on the second side are

the 1st term of $\frac{1}{n}\beta^n(y-\beta)^n=\frac{1}{n}\beta^n y^n$

.. 2nd ..... $\frac{1}{n-1}\beta^{n-1}(y-\beta)^{n-1}=-\frac{1}{n-1}(n-1)y^{n-2}\beta^n$

.. 3rd ... .. $\frac{1}{n-2}\beta^{n-2}(y-\beta)^{n-2}=+\frac{1}{n-2}\frac{(n-2)(n-3)}{1.2}y^{n-4}\beta^n$

.....

.. $(v+1)^{\text{th}}$ term of $\frac{1}{n-v}\beta^{n-v}(y-\beta)^{n-v}$

$$=(-)^v\frac{1}{n-v}\frac{(n-v)(n-v-1)..(n-2v+1)}{1.2....v}y^{n-2v}\beta^n.$$

Hence the coefficient of β^n on the second side is

$$\frac{1}{n}y^n-y^{n-2}+\frac{n-3}{2}y^{n-4}-\frac{(n-4)(n-5)}{2.3}y^{n-6}+etc.\ldots\ldots\ldots(2)$$

Multiplying (1) and (2) by n and equating the results, we get

$$x^n+\frac{1}{x^n}=y^n-ny^{n-2}+\frac{n(n-3)}{1.2}y^{n-4}-\frac{n(n-4)(n-5)}{1.2.3}y^{n-6}+..$$

the general term being

$$(-)^v\frac{n(n-v-1)(n-v-2)....(n-2v+1)}{1.2.3....v}y^{n-2v}.$$

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[FIFTH SOLUTION. *Mr. Nicholas Smyth, Galway.*]

Let $y = x+v$, and $z=xv$. Then by the well known process of deriving the sum of the n^{th} powers of two quantities in terms of their sum and product, we have

$$x^n+v^n=y^n-ny^{n-2}z+\frac{n(n-3)}{2}y^{n-4}z^2+\ldots.$$

Substituting in this equation the given values of v and z, viz., $\frac{1}{x}$ and 1, we get the enunciated theorem.

Good solutions were also received from A.D.B., Bristol, and C. E. S. Baynes, Shooter's Hill School.

LIX. *By* γ.

If the outer envelope be removed from all the faces of a complete pile of shot, give an expression for the number of shot left in each kind of pile; and if the outer border be removed from each kind of course, give an expression for the number of shot left.

[FIRST SOLUTION. *By the Proposer.*]

When the border of balls is removed from the n^{th} course of a pile of balls, the remaining balls of that course constitute, respectively,

In the triangular pile, the $(n-3)^{th}$ course,
.... square..... $(n-2)^{th}$ ..... ,
.... rectangular..... $(n-2)^{th}$ .....

For in such case there are removed from each side of the triangular course, $(n-1)$ balls, and hence $3n-3$ from all the sides. We have therefore for the number left,

$$\frac{n(n+1)}{2}-(3n-3)=\frac{n^2+n-6n+6}{2}=\frac{(n-2)(n-3)}{2},$$

the number in a course of $n-3$ lines.

Again, there are removed $n-1$ balls from each side of the square course, and consequently $4n-4$ in all. Whence

$$n^2-(4n-4)=(n-2)^2=\text{course of } n-2 \text{ in a side.}$$

In a rectangular course of n and $n+m$ balls,

$$2\{(m+n-1)+(n-1)\}=\text{balls removed,}$$

$$\therefore\ n(m+n)-2(m+2n-2)=mn+n^2-2m-4n+4$$
$$=(m+n-2)(n-2),$$

the same as a course in which $n-2$ is put for n.

This completes the investigation of one part of the problem.

If the entire covering of balls of a pile of n courses be removed, there will be left,

In the triangular one, a pile of $n-4$ courses,
.... square ....., ...$n-2$ .....,
.... rectangular .., ...$n-3$ .....

Triangular pile. It will be obvious that

$$\frac{4(n-3)(n-2)}{2}=2(n-3)(n-2)=\text{balls in four faces not including the edges,}$$

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$3(n-1)+3(n-2)+1$ = balls in the edges,

$\therefore\ 2(n-3(n-2)+3(2n-3)+1 = 2n^2-4n+4$ = balls taken away.

Hence we have for the number left

$$\frac{1}{6}n(n+1)(n+2) - (2n^2-4n+4) = \frac{1}{6}(n^3-9n^2+26n-24)$$

$$= \frac{1}{6}(n-4)\{(n-4)+1\}\{(n-4)+2\},$$

a triangular pile of $n-4$ courses.

Square pile. Here also it is obvious that

$(n-2)^2$ = bottom course, not including border,

$\frac{4(n-3)(n-2)}{2} = 2(n-3)(n-2)$ = four triangular faces, not including edges,

$4(n-1)+4(n-2)+1$ = edges and border of base ;

$\therefore\ (n-2)^2+2(n-3)(n-2)+4(2n-3)+1 = 3n^2-6n+5$, the balls taken away. Therefore

$$\frac{1}{6}n(n+1)(2n+1) - (3n^2-6n+5) = \frac{1}{6}(2n^3-15n^2+37n-30)$$

$$= \frac{1}{6}(n-3)\{(n-3)+1\}\{2(n-3)+1\},$$

a square pile of $n-3$ courses.

Rectangular pile. In this pile n is diminished by 3 as in the square pile, and m is unaffected. Hence the expression is

$$\frac{1}{6}(n-3)(n-2)\{3m+2(n-3)+1\}.$$

[Second Solution. *Mr. R. Harley, Northwich, Cheshire.*]

A little consideration will shew that the number of shot remaining when the outer envelope is removed from the *triangular pile* is equal to the number that is left when the *four* bottom strata, or courses, are taken away ; and the number remaining when the outer border is removed from a triangular *course* is equal to the number left when *three* rows are removed. In the *square* and *rectangular* piles, the corresponding courses and rows remaining, are $n-3$ and $n-2$.

Hence we have

$$1+3+6+10+\ldots\text{to } (n-4) \text{ terms} = \frac{(n-2)(n-3)(n-4)}{1.2.3},$$

the number left in the triangular pile when the outer envelope is removed.

And writing $(n-3)$ for n in $\frac{1}{2}n(n+1)$, we have $\frac{1}{2}(n-2)(n-3)$, the number left in the triangular course when the border is removed.

The corresponding expressions for the square and rectangular piles, are

$$\frac{(n-3)(n-2)(2n-5)}{1.2.3}, \quad (n-2)^2;$$

and $\dfrac{(n-3)(n-2)\{3m+2(n-3)+1\}}{1.2.3}$, $(n-2)(m+n-2)$.

Messrs. S. Bills, Hawton; John Laws, Newcastle; Nicholas Smyth, Galway, also sent solutions.

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LX. *Mr. J. W. Elliott, Greatham.*

Find the moment of inertia of a regular polygon of n sides revolving about an axis through its centre, and making an angle α with its plane.

[FIRST SOLUTION. *Mr. G. W. Hearn, R.M. College, Sandhurst.*]

1. To find the moment of inertia of an isosceles triangle about principal axes through the vertex.

Let the axis of the triangle be the axis of x, e the semi-angle at the vertex, a the length of the axis, b the base, and $A = \frac{1}{2}ab =$ the area. Then the moment of inertia about the axis of x is

$$= \int\int y^2 dy dx = \tfrac{2}{3}\int y^3 dx\,;\ \text{but } y = x\tan e\,;\ \text{therefore}$$

$$\text{moment} = \tfrac{2}{3}\tan^3 e \int x^3 dx = \tfrac{1}{4} A a^2 \tan^2 e.$$

Similarly the moment about the axis of y is $= \int\int x^2 dy dx$

$$= 2\int x^2 y dx = 2\tan e \int x^3 dx = \tfrac{1}{2}\tan e . a^4 = \tfrac{1}{2} A a^2.$$

2. Let a line be drawn through the vertex still in the plane of the triangle, and making an angle θ with the axis of x.

Then by the transformation of co-ordinates we have the moment of inertia about this line

$$= \Sigma m(y\cos\theta - x\sin\theta)^2 = \Sigma m y^2 . \cos^2\theta + \Sigma m x^2 . \sin^2\theta,$$

since $\Sigma m x y = 0$, the axes of reference being principal axes through the vertex. Putting therefore $\frac{1}{6} A a^2 \tan^2 e$ for $\Sigma m y^2$, and $\frac{1}{2} A a^2$ for $\Sigma m x^2$, the above moment

$$= \tfrac{1}{6} A a^2 \tan^2 e \cos^2\theta + \tfrac{1}{2} A a^2 \sin^2\theta$$

$$= \tfrac{1}{12} A a^2 \{3 + \tan^2 e - (3 - \tan^2 e)\cos 2\theta\}.$$

Now a regular polygon of n sides consists of n isosceles triangles, and the moment of inertia of the figure is equal to the sum of the moments of inertia of the several triangles. Making therefore $e = \frac{\pi}{n}$, and putting $\theta + r.\frac{2\pi}{n}$ for θ, and then taking the finite integral $\Sigma_r \cos\left(2\theta + r\frac{4\pi}{n}\right)$ from $r = n$ to $r = 0$, which is found to be zero; we obtain for the moment of inertia of the polygon round *any* axis through its centre and in its own plane

$$\mu = \frac{1}{12} n A a^2 \left(3 + \tan^2\frac{\pi}{n}\right).$$

3. Let an axis pass through its centre and be perpendicular to its plane. Then the moment of inertia about this axis is $= \Sigma m(x^2 + y^2) = 2\mu$.

We have now the moment of inertia about the three principal axes through the centre, namely, μ about each in the plane of the figure, and 2μ about that at right angles to that plane. Taking therefore θ and ϕ for the angles at which the given axis is inclined to those in the plane of the figure, we have

$$\cos^2\theta + \cos^2\phi = \cos^2 a\,;$$

therefore the required moment of inertia

$$= 2\mu\sin^2 a + \mu\cos^2\theta + \mu\cos^2\phi$$

$$= 2\mu\sin^2 a + \mu\cos^2 a = \mu(1 + \sin^2 a),$$

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and therefore if $nA = M$, the mass of the polygon (density $= 1$) ;

$$\text{the moment} = \frac{1}{12} Ma^2\left(3 + \tan^2\frac{\pi}{n}\right)(1+\sin^2\alpha)$$

$$= \frac{1}{12n} M^2\left(3\cot\frac{\pi}{n} + \tan\frac{\pi}{n}\right)(1+\sin^2\alpha).$$

[SECOND SOLUTION. *Mr. Thomas Weddle, Newcastle.*]

1. To find the moment of inertia of a triangle ABC about its base BC.

Let the perpendicular from the vertex upon the base $= p$, the base $= a$, and the distance of a line parallel to the base $= x$; then by similar triangles the length of this variable parallel intercepted by the sides of the triangle is $= \frac{a}{p}(p-x)$; hence the moment of the element corresponding to this line $= \frac{a}{p}(px^2-x^3)\delta x$. Integrating this between the limits $x = 0$, and $x = p$, we get

$$Mk^2 = \frac{ap^3}{12} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

2. To find the moment of inertia of an isosceles triangle AOB about an axis OG in its own plane, drawn through the vertex O.

Let AB produced meet OG in G; draw AI and BH perpendicular to OG. Put $AO = BO = a$, angle $BOG = \alpha$, and angle $AOB = \beta$; then angle $OBG = \frac{\pi}{2} + \frac{1}{2}\beta$, and $G = \frac{\pi}{2} - (\alpha + \frac{1}{2}\beta)$; wherefore $OG = \frac{a.\cos\frac{1}{2}\beta}{\cos(\alpha + \frac{1}{2}\beta)}$, $AI = a\sin(\alpha+\beta)$ and $BH = a\sin\alpha$.

Hence, taking the moment of inertia about OG, we have by (1)

$$Mk^2 = \text{moment of } \triangle AOB = \text{moment of } \triangle AOG - \text{moment of } \triangle BOG$$

$$= \frac{OG}{12}(AI^3 - BH^3) = \frac{OG(AI - BH)}{12}(AI^2 + AI.BH + BH^2)$$

$$= \frac{a^4\sin\frac{1}{2}\beta.\cos\frac{1}{2}\beta}{6}\left\{\sin^2(\alpha+\beta) + \sin(\alpha+\beta).\sin\alpha + \sin^2\alpha\right\},$$

$$\text{or } Mk^2 = \frac{a^4}{24}.\sin\beta.\{2+\cos\beta-(1+2\cos\beta)\cos(2\alpha+\beta)\} \quad \ldots\ldots\ldots\ldots (2)$$

3. To find the moment of inertia of a regular polygon $A_1A_2..A_n$ of n sides about an axis OB in its plane, drawn through the centre O.

If $OA_1, OA_2..OA_n$ be joined, the polygon will be divided into n isosceles triangles, and the sum of the moments of these triangles will be the moment required. The angles $A_1OA_2, A_2OA_3..A_n OA_1$ are each $= \frac{2\pi}{n}$; if, therefore, we reckon the angles one way round, and put angle $A_1OB = \alpha$, we shall have

$$A_2OB = \alpha + \frac{2\pi}{n},\ A_3OB = \alpha + 2\frac{2\pi}{n},\ldots A_nOB = \alpha + (n-1)\frac{2\pi}{n}.$$

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Hence if, in (2), we substitute $\frac{2\pi}{n}$ for β, and $a + m.\frac{2\pi}{n}$ for a, and take the sum for all whole values of m between 0 and $n-1$, we shall get the moment of inertia of the polygon $A_1A_2 \ldots A_n$ about OB; hence $Mk^2 =$

$$\frac{a^4}{24}\sin\frac{2\pi}{n}\left[\left(2+\cos\frac{2\pi}{n}\right).n-\left(1+2\cos\frac{2\pi}{n}\right)\Sigma_0^{n-1}\cos\left\{2a+(2m+1)\frac{2\pi}{n}\right\}\right]$$

where $a = OA_1 = OA_2 \ldots = OA_n$. Moreover (Hind's Trig. 4th ed. p. 272.)

$$\Sigma_0^{n-1}\cos(a+m.\delta) = \frac{\cos\{a+\frac{1}{2}(n-1)\delta\}.\sin\frac{1}{2}n\delta}{\sin\frac{1}{2}\delta};$$

$$\therefore \Sigma_0^{n-1}\cos\left\{2a+(2m+1)\frac{2\pi}{n}\right\} = 0.$$

Hence since a disappears, the moment of inertia of a regular polygon of n sides about *any* axis in its plane and passing through its centre, is

$$Mk^2 = \frac{na^4}{12}.\sin\frac{2\pi}{n}.\left\{2+\cos\frac{2\pi}{n}\right\} \ldots\ldots\ldots\ldots(3)$$

Hence also (Earnshaw's Dynamics, p. 174), the moment about the axis through the centre of the polygon perpendicular to its plane, is

$$Mk^2 = \frac{na^4}{12}\sin\frac{2\pi}{n}.\left\{2+\cos\frac{2\pi}{n}\right\} \ldots\ldots\ldots\ldots(4)$$

4. To find the moment of inertia of a regular polygon of n sides about an axis OB through its centre O, which is inclined at an angle a to its plane.

Take the centre for the origin of rectangular axes, the axis of z being perpendicular to the plane of the polygon. Let α', β', γ', be the angles which the axis OB makes with the axes x, y, z and A, B, C the moments of inertia about these axes. Hence, (Poisson's Mécanique, art. 375.)

$$Mk^2 = A\cos^2\alpha' + B\cos^2\beta' + C\cos^2\gamma'.$$

Moreover (3, 4), $A = B = \frac{1}{2}C = \frac{na^4}{24}.\sin\frac{2\pi}{n}\left\{2+\cos\frac{2\pi}{n}\right\}$; hence recollecting that $\cos^2\alpha' + \cos^2\beta' + \cos^2\gamma' = 1$, and $\cos^2\gamma' = \sin^2 a$, we finally have

$$Mk^2 = \frac{na^4}{24}\sin\frac{2\pi}{n}.\left\{2+\cos\frac{2\pi}{n}\right\}.\{\cos^2\alpha'+\cos^2\beta'+2\cos^2\gamma'\}.$$

$$= \frac{na^4}{24}.\sin\frac{2\pi}{n}.\left\{2+\cos\frac{2\pi}{n}\right\}.(1+\sin^2 a) \ldots\ldots\ldots\ldots(5)$$

[THIRD SOLUTION. *Lieut. St. Andrew St. John, R.E.; Messrs. J. W. Elliott; John Laws; and R. H. Wright, London.*]

Let AX, AY, AZ be rectangular axes originating at A the centre of any regular polygon, and AB any other axis making with the principal axes the angles α', β', γ'. Then, if A, B, C be the three moments of inertia of the polygon with respect to the axes of x, y, z, the moment of inertia with respect to the axis AB will be

$$Mk^2 = A\cos^2\alpha' + B\cos^2\beta' + C\cos^2\gamma'.$$

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In the case of any regular polygon we have

$$A = B = \tfrac{1}{2}C.$$

Let one of the sides of the polygon $= a$, and the radius of the circumscribed circle $= R$; then (Walton's Mechanical Problems, page 295,) the moment of inertia of one of the equal isosceles triangles (composing the polygon) about the axis of $z = \frac{1}{12}\Delta(6R^2 - a^2)$;

$$\therefore \quad C = \frac{1}{12} n\Delta(6R^2 - a^2) = \frac{1}{12} M(6R^2 - a^2),$$

n denoting the number of sides of the polygon;

$$\text{but} \quad R = \frac{a}{2\sin\frac{\pi}{n}}; \quad \therefore C = \frac{a^2M}{12} \cdot \frac{2 + \cos\frac{2\pi}{n}}{1 - \cos\frac{2\pi}{n}}.$$

$$\text{Hence} \quad Mk^2 = \frac{a^2M}{24} \cdot \frac{2 + \cos\frac{2\pi}{n}}{1 - \cos\frac{2\pi}{n}} (1 + \sin^2 a).$$

The solutions of Messrs. H. J. Hose, Trinity Square, Southwark, and Nicholas Smyth, Galway, are omitted with much regret.

MATHEMATICAL EXERCISES—*(continued.)*

67. *Mr. Matthew Collins, Limerick.*

Determine geometrically an arc of a given circle such that the excess of its chord above the chord of half the arc may be a maximum.

68. *Mr. John Riddle, Nautical School, Greenwich Hospital.*

An observer stands in such a position as to see three angles of a square tower standing on a horizontal plane, and he observes the angles of elevation of the three upper corners to be, for the nearest $\alpha°$, and for the other two $\beta°$ and $\gamma°$; and his nearest distance from the tower is m feet. Required the length of the side of the square, and the height of the tower.

69. *Mr. J. W. Elliott, Greatham.*

If from a point P in a conic section, a tangent PO be drawn to meet the axis major AB in O, and the chords AP, BP to meet a perpendicular through O in Q, Q': then QO = OQ'.

70. *Mr. Thomas Cotterill, Panton Square, Haymarket.*

If K be the area of a triangle ABC whose sides are a, b, c; R the radius of the circumscribed circle, and α, β, γ, points in the sides a, b, c, or in the sides produced; then K' being the area of the triangle $\alpha\beta\gamma$, we shall have

$$\frac{A\beta \cdot C\alpha \cdot B\gamma + C\beta \cdot A\gamma \cdot B\alpha}{K'} = \frac{abc}{K} = 4R.$$

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71. *By the same Gentleman.*

If the triangle K′ taken as in last exercise be co-polar with the triangle K, S being the pole, and K_a, K_b, K_c be the areas of the triangles BCS, CAS, ABS respectively; then

$$\frac{2}{K'} + \frac{1}{K} = \frac{1}{K_a} + \frac{1}{K_b} + \frac{1}{K_c}.$$

72. *Mr. Rutherford.*

If three equal forces be represented in magnitude and direction by the three lines drawn from the angular points of a plane triangle to the centre of the circumscribed circle, their resultant will be represented in magnitude and direction by the straight line which joins the centre of the circumscribed circle and the point of intersection of the perpendiculars from the angles on the opposite sides.

73. *Mr. Fenwick.*

Let the diagonals AC, BD of a parallelogram ABCD intersect in O, and through O draw EF, GH parallel respectively to the sides AB, BC. In EF take any point E, and let lines EAH, EDG meet GH in H and G. Then drawing the lines HB, GC, they will tend to a point F in EF, and EFGH will be a parallelogram. The property also holds for other lines through the point O, but not for *any* two lines; specify the failing cases.

74. *Mr. Thomas Weddle, Newcastle-on-Tyne.*

Let lines be drawn from the angles of a triangle through two points and terminating in the opposite sides, then if the extremities of each set of lines be joined, two inscribed co-polar triangles will be formed. Also from the angular points of the primitive triangle let lines be drawn parallel to the corresponding sides of these triangles, and intersecting those of the other: it is required to prove that the three points of intersection range in a straight line.

75. *By P. K.*

Integrating by parts the formula $\int e^x\,dx = e^x$, it is required to shew that

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \text{etc.}$$

76. *Mr. Fenwick.*

All surfaces of the second order described through the angular points of a given parallelopiped have the same centre, and the diameters of each drawn parallel to the edges of the parallelopiped are conjugate diameters.

77. *Mr. James Reid, Prior's Salford.*

If in each of the straight lines which form the edges of a tetrahedron any two points be taken equally distant from the middle points of the edges, the twelve points lie in the same conical surface.

78. *Mr. J. F. Heather, M. A., R. M. Academy.*

An elastic ring being placed upon a sphere which revolves round a vertical axis, and the diameters of the ring and sphere being given, determine the angular velocity that the ring may slip completely over the sphere.

79. *Professor Hearn, R. M. College, Sandhurst.*

On a rough board inclined at a given angle to the horizon a rough ball is projected horizontally: required its path and a confirmation of the result from actual experiment.

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THE MATHEMATICIAN.

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ON PROPOSITIONS IN PLANO AND THEIR ANALOGOUS PROPERTIES IN THREE DIMENSIONS.

[*Professor Hearn, Royal Military College.*]

I propose to discuss the question—Does there always exist a problem in three dimensions analogous to a given problem in plano?

Before, however, I enter upon this discussion, I would remark in reference to a paper in Number 1, Vol. II, of the Mathematician, by Mr. Fenwick, entitled "A property relative to surfaces which pass through the same eight points in space," that the enunciation of the question in the Diary to which Mr. Fenwick refers, ought to have been the following:

"An indefinite number of surfaces of the second order may pass through eight points in space. Let diameters be drawn parallel to a given line, then will the diametral planes which are conjugate to these diameters all pass through the same line."

I now give my solution to the problem in its amended form.

Let $u = 0$, $v = 0$, be two equations of the second order in x and y and z, representing two of the system of surfaces. Then any of the other surfaces will be defined by

$$u + \lambda v = 0,$$

in which λ is arbitrary.

Let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be equations of the given direction, then the equation to a plane conjugate to a diameter parallel to this direction is

$$l\frac{du}{dx} + m\frac{du}{dy} + n\frac{du}{dz} + \lambda\left\{l\frac{dv}{dx} + m\frac{dv}{dy} + n\frac{dv}{dz}\right\} = 0,$$

which represents a plane passing through the line of intersection of the two planes whose equations are

$$l\frac{du}{dx} + m\frac{du}{dy} + n\frac{du}{dz} = 0,$$

$$l\frac{dv}{dx} + m\frac{dv}{dy} + n\frac{dv}{dz} = 0.$$

Let us now consider the following problem.

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To find the locus of the diameters conjugate to sections which are all parallel to a given plane.

Let $lx + my + nz = 0$, be the given plane. Then

$$\frac{1}{l}\frac{du}{dx} + \frac{\lambda}{l}\frac{dv}{dx} = \frac{1}{m}\frac{du}{dy} + \frac{\lambda}{m}\frac{dv}{dy} = \frac{1}{n}\frac{du}{dz} + \frac{\lambda}{n}\frac{dv}{dz},$$

are the equations to the conjugate diameter.

Eliminate λ, and we have

$$l\left\{\frac{du}{dy}\frac{dv}{dz} - \frac{dv}{dy}\frac{du}{dz}\right\} + m\left\{\frac{du}{dz}\frac{dv}{dx} - \frac{du}{dx}\frac{dv}{dz}\right\} + n\left\{\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}\right\} = 0,$$

for the locus sought, which is therefore a hyperboloid of one sheet. This equation is always satisfied by the two equations

$$\frac{du}{dx} \div \frac{dv}{dx} = \frac{du}{dy} \div \frac{dv}{dy} = \frac{du}{dz} \div \frac{dv}{dz},$$

whatever be the values of l, m, and n. Consequently all the hyperboloids intersect in the same curve, which is of course the centres of the surfaces $u + \lambda v = 0$. *(Vide 65 Quest.)*

Under particular conditions the hyperboloid becomes a cone. Mr. Fenwick has very ably discussed some of these cases in the paper alluded to, but he seems by his note at the end to have considered that he had established the *non-existence* of a theorem in three dimensions analogous to that in plano enunciated at the commencement of his paper. The problem here given, however, which was that really intended to have been proposed in the Diary, is a strict analogy.

I will now consider the question, "Does there always exist a problem in three dimensions analogical to a given problem in plano?"

Every plane problem may, in one sense, be regarded as a solid problem, for every straight line may be considered as the intersection of a plane (at right angles suppose) to the plane of the diagram, and every curve as the intersection of an orthographic cylinder. Now, instead of such surfaces being orthographic, suppose them all to converge to a point at a finite distance from the plane of the diagram. Take this point as a centre, and describe a sphere cutting the geometrical system, and we have a spherical analogy to the proposed plane problem. Generalize one step further. Instead of making the surfaces and planes terminate in a point, suppose them variously disposed. Still there must exist some law of configuration. In other words, since every curve may be considered as composed of an infinite number of infinitely small straight lines, and every surface as composed of an infinite number of planes, the analytical expression of a plane problem may be written $f(ax + by + 1) = 0$, that is, every plane problem requires the combination of a finite or infinite number of equations of the form $ax + by + 1 = 0$. Similarly, the analytical expression of a solid problem is $\Gamma(ax + by + cz + 1) = 0$. The former can in every instance be generalized into the latter, though the actual performance of the generalization in some instances may involve considerable difficulty. The maxim

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'De non apparentibus et non existentibus eadem est ratio,' may do very well in law, but certainly not in mathematics.

Sometimes the generalized problem in three dimensions is self evident, and when so, the deduction of the plane problem from it is at once an easy and elegant mode of solution. As an illustration of this remark, I refer to Woolhouse's demonstration of the properties of converging triangles, (Lady's and Gentleman's Diary, 1842), by a mode of investigation which I discovered about the same time as he did. As a further illustration, I here present you with a demonstration of the theorem.

"If any two triangles be inscribed in a conic section, they will circumscribe another conic section."

Since innumerable surfaces of the second order can be made to pass through eight points arbitrarily situated in space, or can be made to touch eight planes arbitrarily situated, it follows that if any two tetrahedra be inscribed in a surface of the second order, their planes will be tangential to another surface of the second order. Now, suppose two tetrahedra inscribed in a cone of the second order, their planes must touch some surface of the second order. Let now one of the angular points of each tetrahedron approach to and ultimately coincide with the vertex of the cone. Still the eight faces of the tetrahedra are tangential to some surface of the second order. Let a cone having the same vertex as that already described envelope this surface—then six of the planes of the tetrahedra (namely, those through the common vertex of the cones) are necessarily tangential to the enveloping cone, and *this cone is necessarily one of the second order*. Now let the geometrical system of two cones and six planes be cut by another plane, and the section will be two curves of the second order, with two triangles inscribed in the one, also circumscribing the other.

Instead of cutting the system by a plane, let it be cut by a sphere whose centre is the common vertex, and we have a demonstration of the analogical spherical problem.

Royal Military College, Nov. 20, 1845.

PROPERTIES OF CONIC SECTIONS AND THEIR INSCRIBED AND CIRCUMSCRIBED PARALLELOGRAMS.

[*Mr. Fenwick.*]

The following properties relative to curves of the second order and their inscribed and circumscribed parallelograms, which are new so far as I know, may be interesting to some of our readers. Two of them it will be seen also hold in *space*.

Take the adjacent sides AB, AD, of a parallelogram ABCD, for axes of coordinates, and put $AB = m$, $AD = n$; then (m, n) will denote the point C.

Let a conic section which circumscribes ABCD be

$$y^2 + 2bxy + cx^2 + 2dy + 2ex = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Then since (m, o), (o, n), and (m, n), are points in this conic section, we have

$$cm^2 + 2em = 0 \ldots\ldots (2) \qquad n^2 + 2dn = 0 \ldots\ldots\ldots (3)$$

$$n^2 + 2bmn + cm^2 + 2dn + 2em = 0 \ldots\ldots\ldots\ldots\ldots (4)$$

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Substituting (2) and (3) in (4), b vanishes; the equation (1) therefore becomes

$$y^2 + cx^2 + 2dy + 2ex = 0.$$

Remove the origin to the intersection of the diagonals, by writing $X + \frac{1}{2}m$, for x, and $Y + \frac{1}{2}n$, for y. Then we have

$$(Y + \tfrac{1}{2}n)^2 + c(X + \tfrac{1}{2}m)^2 + 2d(Y + \tfrac{1}{2}n) + 2e(X + \tfrac{1}{2}m) = 0,$$

or, by (2) and (3) and obvious reductions,

$$Y^2 + cX^2 + f = 0 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where $c = -\dfrac{2e}{m}$, and $f = -\frac{1}{4}n^2 + \frac{1}{2}em$, e being arbitrary.

This is the equation of a curve of the second order of which the new origin is the centre, and the axes which are parallel to the sides of the parallelogram, are conjugate diameters. Innumerable curves of the second order (central ones) are evidently represented by (5), as it involves an arbitrary quantity e.

Hence the following properties:

(*a*). *All conic sections described through the angular points of a given parallelogram have the same centre.*

(*b*). *The diameters of each of these conic sections drawn parallel to the sides of the parallelogram are conjugate diameters.*

Again, tangents to (5) at the points A, B, C, D (the axes of reference being the new axes through the point O, the intersection of the diagonals AC, BD), are

$$ny + mcx - 2f = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6)$$
$$ny - mcx - 2f = 0 \ldots\ldots\ldots\ldots\ldots\ldots (7)$$
$$ny + mcx + 2f = 0 \ldots\ldots\ldots\ldots\ldots\ldots (8)$$
$$ny - mcx + 2f = 0 \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Now for all values of c (c being a function of e, as we have seen), (6) is parallel to (8), and (7) to (9); the tangents therefore in every case form a parallelogram. And since for $x = 0$, (6) and (7) give the same values for y, it follows that the tangents at A and B intersect in the axis of y. In a similar way it may be shewn that the other intersecting tangents meet in the axes of coordinates, which are conjugate diameters. We have therefore the additional properties:

(*c*). *If conic sections be described through the angular points of a given parallelogram; then, the tangents to each through the angular points will form a parallelogram.*

(*d*). *The diagonals of the circumscribed parallelograms will lie on common conjugate diameters that are parallel to the sides.*

The following property is also readily deduced from the preceding:

(*e*). *About the same parallelogram there can be described an indefinite number of parallelograms, the diagonals of which all lie on lines parallel to the sides* AB, BC, *drawn through the point* O, *the intersection of the diagonals* AC, BD.

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We will next shew by transforming the coordinate axes, that there are no other conjugate diameters common to all the curves (5).

As a preliminary step, we shall refer all the curves which pass through the points A, B, C, D, to the diagonals BD, AC, as axes of coordinates.

Proceeding as in the preceding part for finding (5), we readily get for all the conic sections which circumscribe ABCD, the equation

$$h^2y^2 - 2bh^2k^2xy + k^2x^2 - h^2k^2 = 0,$$

of which $h = \text{BO}$, $k = \text{CO}$, and b is arbitrary.

Next transform the axes (OB, OC) by means of the expressions

$$x = \frac{\text{X}\sin(\omega - \theta) + \text{Y}\sin(\omega - \phi)}{\sin\omega}, \quad y = \frac{\text{X}\sin\theta + \text{Y}\sin\phi}{\sin\omega},$$

ω being the angle contained by the diagonals BD, AC, and θ, ϕ, the angles which the new axes make with BD. The coefficient of XY of the transformed equation is

$$\frac{2}{\sin^2\omega}\left\{h^2\sin\theta\sin\phi + k^2\sin(\omega - \theta)\sin(\omega - \phi) - h^2k^2bQ\right\},$$

where $$Q = \sin(\omega - \theta)\sin\phi + \sin(\omega - \phi)\sin\theta.$$

Equating this coefficient to zero, the resulting equation is satisfied, whatever be the value of b, by the equations

$$\sin(\omega - \theta)\sin\phi + \sin(\omega - \phi)\sin\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

$$h^2\sin\theta\sin\phi + k^2\sin(\omega - \theta)\sin(\omega - \phi) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

From these we get

$\frac{\sin\theta}{\sin(\omega - \theta)} = \frac{k}{h}$, and $\frac{\sin\phi}{\sin(\omega - \phi)} = -\frac{k}{h}$; and hence the equations of the new axes in reference to the original axes BD, AC, are

$$y = \frac{\sin\theta}{\sin(\omega - \theta)}x = \frac{k}{h}x, \text{ and } y = \frac{\sin\phi}{\sin(\omega - \phi)}x = -\frac{k}{h}x.$$

But the equations of the sides BA, AD, in reference to the axes BD, AC, are

$$-\frac{y}{k} + \frac{x}{h} = 1, \text{ and } -\frac{y}{k} - \frac{x}{h} = 1;$$

or $$y = \frac{k}{h}x - k, \text{ and } y = -\frac{k}{h}x - k.$$

Hence the new axes are parallel to the sides BA, AD.

It will be obvious therefore that there is one pair only of conjugate diameters common to all the curves.

Again, developing (10) and reducing, we get

$$\left.\begin{aligned} &\cot\theta + \cot\phi = 2\cot\omega, \\ \text{or,} \quad &\frac{1}{\tan\theta} + \frac{1}{\tan\phi} = \frac{2}{\tan\omega} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (12)$$

Hence,

(f). *The tangents of the three angles which one of the diagonals of any parallelogram makes with the other diagonal and the sides, are*

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in harmonical progression, or the cotangents of these angles are in arithmetical progression.

The angles, it will be seen, must be taken on the *same side* of the diagonal.

We may now investigate the corresponding properties to (*a*) and (*b*) of surfaces. They may be thus enunciated:

If surfaces of the second order be described through the angular points of a given parallelopiped: then,

(*A*). *All these surfaces will have the same centre.*

(*B*). *The diameters of each drawn parallel to the edges of the parallelopiped will be conjugate diameters.*

Take the axes such that the angular points of the figure may be denoted thus:

$$(ooo), (aoo), (o\beta o), (oo\gamma), (a\beta o), (ao\gamma), (o\beta\gamma), (a\beta\gamma);$$

and let a surface of the second order which circumscribes the figure be

$$x^2+by^2+cx^2+2dxy+2exz+2fyz+2gz+2hy+2kx=0\ldots(13)$$

Wherefore we have the following equations:—

$$ca^2+2ka=0\ldots(14)\quad b\beta^2+2h\beta=0\ldots(15)\quad \gamma^2+2g\gamma=0\ldots(16)$$

$$b\beta^2+ca^2+2da\beta+2h\beta+2ka=0\ldots\ldots\ldots\ldots\ldots(17)$$

$$\gamma^2+ca^2+2ea\gamma+2g\gamma+2ka=0\ldots\ldots\ldots\ldots\ldots(18)$$

$$\gamma^2+b\beta^2+2f\beta\gamma+2g\gamma+2h\beta=0\ldots\ldots\ldots\ldots\ldots(19)$$

$$\gamma^2+b\beta^2+ca^2+2da\beta+2ea\gamma+2f\beta\gamma+2g\gamma+2h\beta+2ka=0\ldots.(20)$$

By (14, 15, 17), $d=0$; (14, 16, 18), $e=0$; (15, 16, 19), $f=0$.

Hence the equation (13) becomes

$$x^2+by^2+cx^2+2gz+2hy+2kx=0.$$

Remove the origin of coordinates to the point $(\frac{1}{2}a, \frac{1}{2}\beta, \frac{1}{2}\gamma)$, by writing $X+\frac{1}{2}a$, for x, $Y+\frac{1}{2}\beta$, for y, $Z+\frac{1}{2}\gamma$, for z. Then we have by the last equation and (14, 15, 16),

$$Z^2+bY^2+cX^2+\tfrac{1}{2}g\gamma+\tfrac{1}{2}h\beta+\tfrac{1}{2}ka=0\ldots\ldots\ldots(21)$$

The new origin is therefore the centre, and the axes which are parallel to the edges of the parallelopiped are conjugate diameters. It will be obvious by the equations which precede, that (21) involves *one arbitrary quantity*; and hence it represents innumerable surfaces of the second order. The theorems (*A*) and (*B*) are therefore proved.

*Cor. If a diameter conjugate to a given plane be drawn to each of the innumerable surfaces of the second order which pass through the angular points of a given parallelopiped, all these diameters will intersect in the same point, the common centre of all the surfaces** (see p. 14, Vol. II., for some analogous properties).

R. M. Academy, March 7, 1846.

* We have not given the corresponding property to this in plano, as it is well known. It indeed holds for curves of the second order which pass through the angular points of *any* quadrilateral.

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ON THE THEORY OF COLLINEATION.

[*Mr. Rutherford.*]

The collinear theory is analogous to the theory of poles and polars, which has been both amply and ably discussed in the pages of the *Mathematician*. In the polar theory, each *point* of the one system of points has a corresponding *line* of the other system,—in the collinear theory, to each *point* of the one system there is a corresponding *point* of the other system, with the condition that if *three points* of the one system are in a straight line, the *three corresponding points* of the other system are also in a straight line; and therefore to each *line* in the one system there is a corresponding *line* in the other system. Like the theory of poles and polars, the theory of collineation may be extensively applied to a variety of researches both in plano and in space, and it will be our object in the following discussion to develope the principle in a simple and intelligible manner, and then to apply it to some interesting inquiries.

I.

Definition.—Two figures, or systems of points and lines which are so related that to each *point* of the one system there is a corresponding *point* of the other system, such that if *three points* of the one system are in a straight line, the *three corresponding points* of the other system are also in a straight line, are called *collinear figures*, or *collinear systems*.

Thus if the points A, B, C; A′, B′, C′, are taken so that the three points A, B, C are in a straight line, and also the three points A′, B′, C′ in another straight line; and also if six other points a, b, c; a', b', c' be taken so far arbitrarily that a, b, c range in a straight line, and also a', b', c' in another straight line; then it is not to be understood that each system of six points so taken forms a collinear system, because each system is not limited to six or any number of points and lines. If however the points A, B, C, A′, B′, C′ and a, b, c, a', b', c' belong to *two* collinear systems, and A, a, B, b, *etc.* be corresponding points; then P the point of intersection of the two lines AA′, BB′ in the one system will correspond to p the point of intersection of the corresponding lines aa', bb', *etc.*, in the other system, in conformity with the preceding definition. In a similar manner the intersections Q, R, S, *etc.* of the straight lines that can be drawn through pairs of the six points A, B, C, A′, B′, C′ will correspond to the intersections q, r, s, *etc.* of the right lines drawn through the corresponding pairs of the six points a, b, c, a', b', c'. The points P, Q, R, *etc.* which have been introtroduced into the one system, and the points corresponding to them p, q, r, *etc.* which have been introduced into the other system, will in a similar manner furnish other corresponding pairs of points, and so on. It is obvious that if three of the points thus introduced as P, Q, R in the one system range in a straight line, the three corresponding introduced points, p, q, r, in the other system, must also range in a straight line. That this is not the case with points *arbitrarily* taken is evident, for the first six points may be taken such that the points of intersection P, Q, R are in one line, and the other six points may be selected in such a manner that the corresponding points of intersection p, q, r are *not* situated in one right line. Such a system of points as the latter would not therefore form a system collinear with the former, inasmuch as the condition specified in the definition of collinear systems is not fulfilled.

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Since two points determine a straight line, it is evident that to a straight line of the one system there corresponds a straight line of the other system, and that a straight line which joins two points of the one system has for its corresponding line, that which joins the corresponding points of the other system.

It will be shewn in the next article that, in general, four points of the one system, and four corresponding points of the other system are sufficient to determine a point of the latter system corresponding to a specified point of the former.

II.

Let xy be the rectangular or oblique coordinates of a point in the one system, and tu those of the corresponding point in the other collinear system, referred to the same or different axes; then if both systems are given, the coordinates tu of the one system will be functions of the coordinates xy of the other; hence if

$$gu + ht + k = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be the equation of a *line* in the system (tu), the equation of a *line* in the system (xy), which is to correspond to (1), will be of the form

$$g(m'y + n'x + p') + h(m''y + n''x + p'') + k(my + nx + 1) = 0 \ldots (2)$$

whence if xy be the coordinates of a *point* in the system (xy), the coordinates of a corresponding *point* in the system (tu) will be

$$u = \frac{m'y + n'x + p'}{my + nx + 1} \text{ and } t = \frac{m''y + n''x + p''}{my + nx + 1}; \ldots\ldots\ldots (3)$$

because the substitution of these values for u and t in eq. (1) produces the corresponding line (2).

It is obvious, therefore, that if the coordinates of *four points* of the one system, and those of the *four corresponding points* of the other system, are known, we shall have *eight* equations of the first degree to determine the values of the eight constants in equations (3).

The values of x and y deduced from (3) will obviously be similar in form to those of t and u, and hence the *relation of collineation* exists mutually between two systems.

If rs be the coordinates of a third system of points and lines which is collinearly related to the system (tu); then we must have

$$u = \frac{f's + g'r + h'}{fs + gr + 1} \text{ and } t = \frac{f''s + g''r + h''}{fs + gr + 1},$$

and therefore also

$$\frac{f's + g'r + h'}{fs + gr + 1} = \frac{m'y + n'x + p'}{my + nx + 1}; \quad \frac{f''s + g''r + h''}{fs + gr + 1} = \frac{m''y + n''x + p''}{my + nx + 1} \ldots (3')$$

From these equations the values of r and s can be determined in terms of x and y, and as these values are similar in form to those of t and u, it follows that the system (xy) is collinearly related to the system (rs). Hence also if each of two systems is collinearly related to a third system, the two systems are themselves collinearly related; and therefore the equations (3') may be employed in order to determine the relative properties which exist in reference to two collinearly related systems.

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III.

The equations (3) of the preceding article admit of simplification. Removing the origin of coordinates in both systems, by writing $t+t'$, $u+u'$, $x+x'$, $y+y'$ for t, u, x, y, in these equations, we have

$$\left.\begin{aligned} u+u' &= \frac{m'y+n'x+m'y'+n'x'+p'}{my+nx+my'+nx'+1} \\ t+t' &= \frac{m''y+n''x+m''y'+n''x'+p''}{my+nx+my'+nx'+1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (u)$$

Now if two corresponding points $x'y'$ and $t'u'$ are taken for origin in each system; then (3)

$$u' = \frac{m'y'+n'x'+p'}{my'+nx'+1}, \quad t' = \frac{m''y'+n''x'+p''}{my'+nx'+1};$$

and if further we put $my'+nx'+1 = q$,

$$\begin{aligned} m'n - mn' &= a; & m'p - mp' &= b; & np' - n'p &= c; \\ m''n - mn'' &= a'; & m''p - mp'' &= b'; & np'' - n''p &= c'; \end{aligned}$$

the expressions (a) reduce to

$$u = \frac{(ax'+b)y-(ay'+c)x}{q(my+nx+q)}; \quad t = \frac{(a'x'+b')y-(a'y'+c')x}{q(my+nx+q)} \quad \ldots\ldots (a')$$

which, when $x=0$ and $y=0$ simultaneously, give also $t=0$, $u=0$.

Transform now the axes, by writing for x and y, the expressions

$$x\cos\theta - y\sin\theta \quad \text{and} \quad x\sin\theta + y\cos\theta;$$

then (a') we have

$$\left.\begin{aligned} u &= \frac{\{(ax'+b)\cos\theta+(ay'+c)\sin\theta\}y+\{(ax'+b)\sin\theta-(ay'+c)\cos\theta\}x}{q\{(m\cos\theta-n\sin\theta)y+(m\sin\theta+n\cos\theta)x+q\}} \\ t &= \frac{\{(a'x'+b')\cos\theta+(a'y'+c')\sin\theta\}y+\{(a'x'+b')\sin\theta-(a'y'+c')\cos\theta\}x}{q\{(m\cos\theta-n\sin\theta)y+(m\sin\theta+n\cos\theta)x+q\}} \end{aligned}\right\} \ldots (\beta)$$

Now from the indeterminateness of x', y', θ, we may assume

$$\left.\begin{aligned} (ax'+b)\cos\theta+(ay'+c)\sin\theta &= (a'x'+b')\sin\theta-(a'y'+c')\cos\theta \\ (ax'+b)\sin\theta-(ay'+c)\cos\theta &= 0 \\ (a'x'+b')\cos\theta+(a'y'+c')\sin\theta &= 0 \end{aligned}\right\} \ldots (\gamma)$$

Dividing equations (γ) by $\cos\theta$, and taking the sum of the last two, gives

$$\begin{aligned} \{a'x'+b'-(ay'+c)\}\tan\theta &= ax'+b+a'y'+c' \\ (ax'+b+a'y'+c')\tan\theta &= -\{a'x'+b'-(ay'+c)\}. \end{aligned}$$

Eliminating $\tan\theta$, we have

$$(ax'+b+a'y'+c')^2+(a'x'+b'-ay'-c)^2 = 0;$$

therefore $ax'+b+a'y'+c'=0$, and $a'x'+b'-ay'-c=0$;

or $$ax'+b = -(a'y'+c'), \quad a'x'+b' = ay'+c \ldots\ldots\ldots (\gamma')$$

The conditions (γ) and (γ') reduce the equations (β) to the forms

$$u = \frac{py}{my+nx+1}, \quad t = \frac{px}{my+nx+1} \ldots\ldots\ldots\ldots\ldots (4)$$

in which the constants m, n, p have, however, different values from those in (3). The eqs. (γ) give also

$$\tan\theta = \frac{ay'+c}{ax'+b} \text{ and } \tan\theta = -\frac{a'x'+b'}{a'y'+c'},$$

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which, by the conditions (γ'), are equal to each other, and therefore the values of θ are separated from each other by *two* right angles.

Hence if a point in the system (xy) is situated in the axis of y, the coordinates of the point corresponding to it in the system (tu) are

$$u = \frac{py}{my+1}, \qquad t = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4')$$

And if a point in the one system is in the axis of x, the coordinates of the point which corresponds to it in the other system, are

$$u = 0, \quad t = \frac{px}{nx+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4'')$$

If now

$$au + bt + c = 0 \ldots\ldots \ldots\ldots\ldots\ldots\ldots (5)$$

be the equation of a *line* in the system (tu), the equation of the line in the system (xy), which is to correspond to (5), will be, by (4)

$$(ap + cm)y + (bp + cn)x + c = 0 \ldots\ldots\ldots (6)$$

Also, to the lines whose equations are

$$t = 0, \quad u = 0,$$

there will correspond the lines whose equations are

$$x = 0, \quad y = 0,$$

that is, the axis of x corresponds to the axis of t, and the axis of y to the axis of u.

The new origin of the system (xy) is termed the *collinear centre*, or the *collinear point* of the system (xy), and the new origin of the system (tu) is called the *collinear centre* of the system (tu).

IV.

From what has been already remarked it is obvious that each of two collinearly related systems has in general only one collinear point, and that the relations between the coordinates of corresponding points can be expressed by equations of the form (4), if the coordinate axes pass through the collinear point so that the axis of t shall correspond to the axis of x, and the axis of u to the axis of y.

Conceive the one system to be placed on the other so that the axes of x and y respectively coincide with those of t and u; then also all the corresponding lines drawn through the *collinear point* (which is now the common origin) coincide with each other, and every two corresponding points are in the same straight line with the collinear point.

Thus if

$$u = ht,$$

be the equation of any straight line of the system (tu), the equation of the corresponding straight line of the system (xy) is, by the equations (4),

$$y = hx,$$

which consequently coincides with the former. If $x'y'$ and $t'u'$ be the coordinates of corresponding points, then the equations of the lines through these points and the origin are

$$y = \frac{y'}{x'}x, \quad \text{and } u = \frac{u'}{t'}t.$$

But, in consequence of the equations (4), we get $\frac{u'}{t'} = \frac{y'}{x'}$, and therefore the lines expressed by these equations are coincident.

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V.

To determine whether in two collinearly situated systems, there are other corresponding points which coincide with each other, we must have $t = x$ and $u = y$. Now these values being put in the equations (4), viz.:

$$u = \frac{py}{my+nx+1} \quad \text{and} \quad t = \frac{px}{my+nx+1},$$

give

$$y(my + nx + 1 - p) = 0, \qquad x(my + nx + 1 - p) = 0.$$

These equations are evidently satisfied when $x = 0, y = 0$, simultaneously, and also when $my+nx+1-p = 0$; hence we conclude that besides both the collinear points, all the points of the straight line

$$my + nx + 1 - p = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

coincide with the points corresponding to them in the system (tu).

This line (7) is named the *collinear axis*.

Hence *each line of the one system intersects its corresponding line of the other collinearly situated system, in the collinear axis.*

VI.

If two straight lines of the one system are parallel to each other, the lines corresponding to them in the other system are, in general, not parallel to each other.

Let

$$u + ht + k = 0, \qquad u + ht + k' = 0 \ldots\ldots \quad \ldots\ldots (a)$$

be the equations of the parallel lines in the system (tu); then the equations of the lines corresponding to them in the collinear system (xy) are (III) eqs. (4)

$$p(y + hx) + k(my + nx + 1) = 0,$$
$$p(y + hx) + k'(my + nx + 1) = 0,$$

which are evidently not parallel lines, except in the case $h = \frac{n}{m}$. These lines intersect in that point in which the two straight lines

$$y + hx = 0, \quad \text{and } my + nx + 1 = 0$$

intersect, since the values of x and y which satisfy the two former equations, equally satisfy the latter set of equations. The equation

$$my + nx + 1 = 0$$

is the equation of a straight line in the system (xy) which is parallel to the collinear axis (7); consequently the locus of the intersections of all the straight lines in the system (xy) which correspond to *parallel lines* in the system (tu) is a straight line *parallel* to the collinear axis, and expressed by the equation

$$my + nx + 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

This straight line is called the *counter axis* of the system (xy).

There is also a straight line in the system (tu) which is called the *counter axis* of the system (tu); that is, a straight line in which all the lines in the system (tu) intersect, that correspond to parallel lines in the system (xy). For the equation $y + hx = 0$, is that of a straight line passing through the *collinear* centre (the origin of coordinates), and it makes the same angle with the *collinear axis* and also with the *counter axis*, as the parallel lines (a) in the system (tu).

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To determine the equation of the counter axis in the system (tu) we must obtain from (4) the values of the coordinates x and y in terms of t and u. The equations (4) give

$$y = -\frac{u}{mu+nt-p}, \quad x = -\frac{t}{mu+nt-p},$$

and consequently by a process analogous to the preceding,

$$mu+nt-p=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

is the equation of the counter axis in the system (tu).

VII.

If in the equations (4) in (III) we put $p=1$; then

$$my+nx=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

is the equation of the collinear axis, which, in this case, passes through the collinear centre. But if $p=-1$, we have

$$t = -\frac{x}{my+nx+1}, \quad u = -\frac{y}{my+nx+1},$$

and hence $$x = -\frac{t}{mu+nt+1}, \quad y = -\frac{u}{mu+nt+1};$$

therefore the equation of the collinear axis, in this case, is

$$my+nx+2=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

If h, k be the coordinates of a point in the system (xy), then the coordinates of a point corresponding to it in the system (tu) are

$$t = -\frac{h}{mk+nh+1}, \quad u = -\frac{k}{mk+nh+1};$$

and if we regard h, k as the coordinates of a point in the system (tu); then the coordinates of the corresponding point in the system (xy) are

$$x = -\frac{h}{mk+nh+1}, \quad y = -\frac{k}{mk+nh+1}.$$

Now the values of t, x are the same in both systems, as well as those of u, y; hence when $p=-1$, it follows that we may interchange any number of points whatever of the one system, with the corresponding points of the other system. Let P, P′ and Q, Q′ be a pair of corresponding points of two collinear and collinearly situated systems for which $p=-1$; then not only will the lines PQ, P′Q′, (being corresponding lines) intersect in the collinear axis; but the straight lines PQ′, P′Q will also intersect in the same axis, in consequence of the points Q, Q′ being interchangeable.

In the case of $p=-1$, the equation of the counter axis in the system (xy) is, by (8),

$$my+nx+1=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

and the equation of the counter axis in the system (tu) is

$$mu+nt+1=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

hence these axes are coincident, because the values of x, t and y, u are the same in both systems.

Again, the equation of the collinear axis is by (11)

$$my+nx+2=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

consequently, *the distance of the collinear axis from the collinear centre is twice the distance of the counter axis from the collinear centre.*

(To be continued.)

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HORNER ON ALGEBRAIC TRANSFORMATION.

(Concluded from page 37.)

40. Having thus endeavoured to shew that some important portions of the Algebraic and Differential Theories are only exemplifications of a general theorem deduced from the principles of algebraic division, I will close these remarks by briefly describing how the basis of both theories may, by the same simple means, be completed.

The reader will have remarked that the more usual differential formulæ are those, whose coefficients occupy diagonal lines in the table of *art.* 33, *etc.*, and consequently in the general table in *art.* 13. The slightest variation from the principle (*art.* 11) on which that table was constructed would have produced the formulæ in question. It was only requisite to stop the division by making $p = m$, one term earlier in each course than in the next preceding. The effect of this alteration may very easily be introduced into the formulæ in *art.* 13, by referring to what was shewn in a preceding article respecting the cotemporary change effected by division, in the unknown variable of one term and the coefficient of the next. For in reversing this effect, it is obvious that any number of the latter terms of a transformee may be removed into a subsequent one without altering its value, if we observe to change the coefficient of the first of the transplanted terms for that which belongs to its new situation.

41. Thus $M_0 x + N_0$ being changed to $M_1 x + N_0$, this may be substituted for the last two terms of the second equation; which will then end in $L_1^2 x x_1 + M_1 x + N_0$. These terms by merely changing L_1 for L_2 may be put for the last three terms of the third equation; and so on. The effect of continuing this process through n transformees is obvious, and may be thus stated:—

If we take the terms which descend diagonally from N_0 changing x_p for x_{p-1}, they constitute a formula $= fx$. The same applies to the diagonal descending from any other absolute. Hence we have

$$A_n x x_1 \ldots x_{n-1} + B_{n-1} x x_1 \ldots x_{n-2} + \ldots + K_3 x x_1 x_2 + L_2 x x_1 + M_1 x + N_0 = fx \quad (1)$$

and generally,

$$A_{m+x} x_m \ldots x_{m+x-1} + B_{m+x-1} x_m \ldots x_{m+x-2} + \ldots + L_{m+2} x_m x_{m+1} + M_{m+1} x_m + N_m = f(x) \ldots (11)$$

42. These equations are substantially the same as the second case of Lemma 5, Book iii, of the Principia, and rank amongst the most fertile theorems of analysis.* The law of successive derivation as selected from the formulæ in *art.* 14 is this:—

A_n	B_{n-1}	K_3	L_2	M_1	N_0
0	$A_n r_n$	$H_4 r_4$	$K_3 r_3$	$L_2 r_2$	$M_1 r_1$
A_{n+1}	B_n	K_4	L_3	M_2	N_1
0	$A_{n+1}(r_{n+1} - r_1)$ ..	$H_5(r_5 - r_1)$	$K_4(r_4 - r_1)$	$L_3(r_3 - r_1)$	$M_2(r_2 - r_1)$
A_{n+2}	B_{n+1}	K_5	L_4	M_3	N_2
0	$A_{n+2}(r_{n+2} - r_2)$ ..	$H_6(r_6 - r_2)$	$K_5(r_5 - r_2)$	$L_4(r_4 - r_2)$	$M_3(r_3 - r_2)$
A_{n+3}	B_{n+2}	K_6	L_5	M_4	N_3
	etc.		*etc.*	*etc.*	

* For an elegant investigation of the first of them, see Mr. Herschel's Appendix to Lacroix, art. 403.

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Here the characteristic feature of continuous approximation, consisting in the united co-operation of all the previous terms in a series, to complete the transformation of the last term, no longer exists.

43. Putting for the sake of conciseness,

$$a_p = r_m - r_{m-p},\ \beta_p = r_{m-1} - r_{m-p-1},\ \gamma_p = r_{m-2} - r_{m-p-2},\ \textit{etc.}$$

we hence collect the following values :—

$$M_m = \frac{N_m - N_{m-1}}{a_1};\quad L_m = \frac{M_m - M_{m-1}}{a_2};\quad K_m = \frac{L_m - L_{m-1}}{a_3};\ \textit{etc.},$$

in which when m becomes $m-1$, $m-2$, *etc.*, a becomes β, γ, *etc.*

44. These expressions, when $r_m = mi = m\Delta x$, become

$$M_m = \frac{\Delta N_{m-1}}{\Delta x};\quad L_m = \frac{\Delta^2 N_{m-2}}{2(\Delta x)^2};\quad K_m = \frac{\Delta^3 N_{m-3}}{2.3(\Delta x)^3};\ \textit{etc.}$$

transforming equation II. (*art.* 41) into

$$fx = N_m + \frac{\Delta N_m}{\Delta x}(x - mi) + \frac{\Delta^2 N_m}{2(\Delta x)^2}\cdot(x - mi)(x - \overline{m+1}.i) + \ldots \quad \text{(III)}$$

a formula which, when $m = 0$, becomes either the ordinary *Equation of Finite Differences* (viz. case 1 of the aforesaid Lemma) or else *Taylor's Theorem*; and this according as i or Δx is equal to 1 or 0.

45. Again, if we proceed to exterminate M, L, *etc.*, from the equations in (43), we have

$$M_m = \frac{N_m}{a_1} - \frac{N_{m-1}}{a_1},$$

$$L_m = \frac{1}{a_2}\left\{\frac{N_m}{a_1} - \frac{N_{m-1}}{a_1} - \frac{N_{m-1}}{\beta_1} + \frac{N_{m-2}}{\beta_1}\right\}$$

$$= \frac{1}{a_2}\left\{\frac{N_m}{a_1} - \frac{(a_1+\beta_1)N_{m-1}}{a_1\beta_1} + \frac{N_{m-2}}{\beta_1}\right\}$$

$$= \frac{N_m}{a_1 a_2} - \frac{N_{m-1}}{a_1\beta_1} + \frac{N_{m-2}}{a_2\beta_1}\ (\text{since } a_1 + \beta_1 = a_2);$$

and so on of the rest. It is unnecessary to continue the process, which can only consist in repetition of similar combinations. The denominator of N_m will obviously be $a_1 a_2 a_3 \ldots$, and the succeeding denominators are derived from these by changing a previous N_m to N_{m-1}, *etc*, and therefore a to β, *etc.*, and prefixing the a which comes next in course; and the multipliers of the numerator are everywhere destroyed in consequence of having $a_{p+1} = a_1 + \beta_p = a_2 + \gamma_{p-1} = \textit{etc.}$ Hence we see that the n^{th} coefficient in retrocession will be

$$A_m = \frac{N_m}{a_1 \ldots a_n} - \frac{N_{m-1}}{a_1 \times \beta_1 \ldots \beta_{n-1}} + \frac{N_{m-2}}{a_2\beta_1 \times \gamma_1 \ldots \gamma_{n-2}} - \frac{N_{m-3}}{a_3\beta_2\gamma_1 \times \delta_1 \ldots \delta_{n-3}} + \ldots \quad \text{(IV.)}$$

In all this, we assume m to be equal to or greater than n.

46. In the case where $r_m = m$, we have likewise

$$a_m = \beta_m = \gamma_m = \ldots = m;$$

and the preceding equation becomes

$$\frac{\Delta^n f(m-n)}{1 \ldots n} = \frac{fm}{1 \ldots n} - \frac{f(m-1)}{1 \ldots (n-1)} + \frac{f(m-2)}{2.1 \ldots (n-2)} - \frac{f(m-3)}{3.2.1(n-3)} + \ldots$$

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or multiplying by $1..n$, it becomes

$$\Delta^n f(m-n) = fm - \frac{n}{1} f(m-1) + \frac{n(n-1)}{1.2} f(m-2) - \frac{n(n-1)(n-2)}{1.2.3} f(m-3) + \ldots.$$

an elegant and well known formula.

47. To these examples I shall only add that of the *Binomial Theorem*, of which we have not made the slightest use in this investigation; and are therefore at liberty to deduce it from any convenient point of our researches. The route of finite differences is preferred, because the materials for completing it occupy the smallest space, and if selected alone from the contents of this paper, would constitute in my opinion the most simple and satisfactory basis for this purpose, that can be laid.

If N_x, or $fx = (1+Q)^x$, we have

$$\Delta fx = (1+Q)^{x+1} - (1+Q)^x = Q(1+Q)^x = Qfx;$$

consequently, $\Delta^2 fx = \Delta\Delta fx = Q\Delta fx = Q^2 fx$;

and generally $\Delta^m fx = Q^m fx = Q^m (1+Q)^x$.

Therefore $\Delta^m f0 = Q^m$; and hence

$$fx = f0 + \Delta f0.x + \frac{1}{2}\Delta^2 f0.xx_1 + \frac{1}{2.3}\Delta^3 f0.xx_1x_2 + ..$$

the simplest case of equation III above, becomes

$$(1+Q)^x = 1 + \frac{x}{1}.Q + \frac{x}{1}.\frac{x-1}{2}Q^2 + \frac{x}{1}.\frac{x-1}{2}.\frac{x-2}{3}Q^3 + ..$$

which is the theorem required. And here x is manifestly unrestricted; since it is the very same quantity, which in the outset of this investigation represented the root of a general equation.

Scholium. In this essay, the principle of transformation by division by unity has been considered only in reference to the divisor $\frac{x_p - (r_m - r_p)}{x_m} = \frac{x_m}{x_m}$. The reader's occasions may lead him to extend this principle to cases which this divisor does not suit. I have found it serviceable particularly in treating reciprocal equations $\left(\text{divisor } \frac{x_1}{x+x^{-1}} = \frac{x_1}{x_1}\right)$, and in decomposition of fractions, and believe it may be made effective in the investigation of quadratic divisors.

To another case also (a case which harmonizes with the latter portion of this inquiry) it applies so happily that I make no apology for introducing the particulars. Let the formula

$$\phi x = ax \pm bx^2 + cx^3 \pm dx^4 + ex^5 \pm ..$$

be divided continually by the fraction whose denominator is x_1 and its numerator $\frac{x}{1 \pm x} = x_1$; the latter x_1 being never made the numerator when the former can be used. The results are

$$\phi x = ax \pm bx^2 + cx^3 \pm dx^4 + ..$$
$$\phi x = ax \pm b_1 xx_1 + c_1 x^2 x_1 \pm d_1 x^3 x_1 + ..$$
$$\phi x = ax_1 \pm b_1 x_1^2 + c_2 x x_1^2 \pm d_2 x^2 x_1^2 + ..$$
etc. *etc.* *etc.*

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the development of the coefficients being

$$\begin{array}{r} a \pm b + c \pm d + \ldots \\ \mp a - b \mp c - \ldots \\ \hline \pm b_1 + c_1 \pm d_1 + \ldots \\ -b_1 \mp c_1 - \ldots \\ \hline c_2 \pm d_2 + \ldots \\ \mp c_2 - \ldots \\ \hline d_3 + \ldots \\ \textit{etc.} \quad \textit{etc.} \end{array}$$

So that $b_1 = \Delta a$, $c_2 = \Delta^2 a$, $d_3 = \Delta^3 a$, *etc.* giving

$$\phi x = \frac{x}{1 \mp x} \cdot a \pm \left(\frac{x}{1 \mp x}\right)^2 \Delta a + \left(\frac{x}{1 \mp x}\right)^3 \Delta^2 a \pm \ldots$$

which are Euler's well known formulæ for increasing the convergency of series.—[*Cal. Diff.*, p. 282, *et seq.*]

PROPOSITIONS ON THE CONIC SECTIONS.

[*Mr. James Dalmahoy, Edinburgh.*]

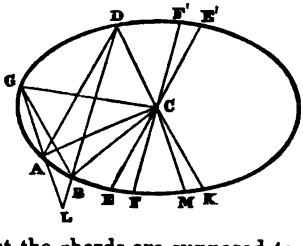
(Continued from p. 63, No. 2., Vol. II.)

PROPOSITION VII. THEOREM.

If from any point in the curve of an ellipse or hyperbola two chords be drawn, and parallel to these respectively two semidiameters; the sector included between the semidiameters is equal to half the sector having for its base the portion of the curve intercepted by the diverging chords.

From D, any point in the curve of an ellipse, draw two chords DA and DB; and respectively parallel to these draw the semidiameters CE and CF: the sector ECF is equal to half the sector ACB.

For produce EC and FC to meet the curve in E', F'; then since AD is parallel to EE' and BD to FF', the sector DCE' = ACE and DCF' = BCF; therefore DCE' — DCF' = ACE — BCF; but DCE' — DCF' = E'CF' = ECF; and ACE — BCF = ACB — ECF; hence ECF = ACB — ECF, and ECF = $\frac{\text{ACB}}{2}$.



In the hyperbola the proof is the same, but the chords are supposed to terminate in opposite branches.

PROP. VIII. THEOREM.

If through any point in the plane of an ellipse or hyperbola two lines be drawn, each meeting the curve in two points, and two semidiameters be drawn parallel respectively to these lines; the sector included between the semidiameters is equal to half the sum or half the difference of the sectors having for their bases the portions of the curve intercepted

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by the two diverging lines, according as the point of divergence of those lines is within or without the curve.

1. Through H, any point within the ellipse, let two chords AHD and BHG be drawn, and parallel to these respectively draw the semidiameters EC and KC; the sector KCE is equal to half the sum of the sectors ACB and DCG.

For join the points B, D, and parallel to BD draw the semidiameter FC; then by the last proposition, the sector KCF is equal to half DCG, and the sector ECF is equal to half ACB; therefore KCF + ECF = ECK $= \frac{ACB+DCG}{2}$.

2. Through L, any point without the ellipse, draw two chords AG and BD, and parallel to these respectively the semidiameters MC and FC: the sector FCM is equal to half the difference of the sectors DCG and ACB.

For draw AD, and parallel to it the semidiameter EC; then by the last proposition, the sector ECM is equal to half DCG, and the sector ECF is equal to half ACB; therefore FCM = ECM − ECF = $\frac{DCG-ACB}{2}$.

PROP. IX. THEOREM.

(Corollary to Desargue's Theorem.)

If a conic section be described about each of the simple quadrilaterals of which a complete quadrilateral is composed, and a transversal be drawn intersecting the three curves and the four sides of the quadrilateral, these intersections will, by Desargue's theorem, furnish three systems of points in involution, and the centres of any two of these systems are conjugate points with respect to the centre of the third system.

Lemma. Let A, B, C, D be any four points lying in a straight line, and let

A B C D

P P′ P″

P, P′, P″ be three points in the same line such that
AP.BP=CP.DP..(1), AP′.CP′=BP′.DP′..(2), AP″.DP″=BP″.CP″..(3), then will

P′P.P″P = AP.BP; PP′.P″P′ = AP′.CP′; PP″.P′P″ = AP″.DP″.

For equation (2) may be written

$$(AP + P'P)(CP - P'P) = (BP + P'P)(DP - P'P);$$

therefore $$P'P = \frac{AP.CP - BP.DP}{(AP-BP)+(DP-CP)}.$$

In like manner (3) may be written

$$(P''P + AP)(P''P - DP) = (P''P + BP)(P''P - CP);$$

$$\therefore P''P = \frac{AP.DP-BP.CP}{(AP-BP)-(DP-CP)}.$$

And if these values of P′P and P″P be multiplied together, the result, after reduction, is

$$P'P.P''P = AP.BP.$$

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In a similar manner, from equations (1) and (3) respectively, we get

$$PP' = \frac{AP'.BP' - CP'.DP'}{(AP'+CP')+(BP'+DP')}; \quad P'P' = \frac{AP'.DP' - BP'.CP'}{(AP'+CP')-(BP'+DP')};$$

hence $$PP'.P'P' = AP'.CP'.$$

Lastly, equations (1) and (2) give respectively

$$PP'' = \frac{AP''.BP'' - CP''.DP''}{(AP''-DP'')+(BP''-CP'')}; \quad P'P'' = \frac{AP''.CP'' - BP''.DP''}{(AP''-DP'')-(BP''-CP'')},$$

$$\therefore \quad PP''.P'P'' = AP''.DP''.$$

Proceeding to the proposition:—Let A, B, C, D be the points of intersection of a transversal with the sides of the complete quadrilateral; EE', FF', GG' (omitted in the diagram) its intersection with the curves described about the three simple quadrilaterals. Let P be the centre of the conjugate points A, B; C, D; E, E': P' the centre of A, C; B, D; F, F': and P'' the centre of A, D; B, C; G, G'. Then since by the theorem these three systems of points are in involution, we get from the first, second, and third systems respectively

$$AP.BP = CP.DP = EP.E'P$$
$$AP'.CP' = BP'.DP' = FP'.F'P'$$
$$AP''.DP'' = BP''.CP'' = GP''.G'P''.$$

But by the lemma

$$P'P.P''P = AP.BP; \quad PP'.P''P' = AP'.CP'; \quad PP''.P'P'' = AP''.DP'';$$

therefore P', P'' are conjugate points with respect to P; P, P'' with respect to P'; and P, P' with respect to P''.

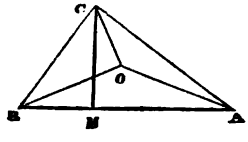
Cor. If one of the diagonals of the quadrilateral pass through one of the centres, the other two diagonals will pass one through each of the other two centres.

MATHEMATICAL NOTE.

[*By a Gentleman Cadet.*]

The fundamental formula for the sine of the sum of two arcs may be very simply deduced in the following manner.

Let ABC be a plane triangle, and draw CM perpendicular to the side AB; then $\sin C = \sin (A + B)$.



Now AB.CM = BM.CM + AM.CM = twice triangle ABC,

and, $AC.BC \sin C$ = twice triangle ABC.

Equating these expressions and dividing by AC.BC, gives

$$\sin C = \frac{CM}{AC}\cdot\frac{BM}{BC} + \frac{AM}{AC}\cdot\frac{CM}{BC},$$

or, $$\sin (A+B) = \sin A \cos B + \cos A \sin B.$$

C.

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COMPUTATION OF TRIGONOMETRICAL TANGENTS AND ANTITANGENTS.

[*Mr. Thomas Weddle, Newcastle.*]

1. In No. 1, Vol. II., of the "Mathematician," I have given a new method of computing logarithms and antilogarithms.* I now proceed to develop a kindred mode of finding the numerical values of tangents and antitangents.†

The following tables will be required, and they are therefore inserted here.

Table I. $\frac{\pi}{180}$ and multiples.

$\frac{\pi}{180}$ =	·017453292519943296
Double =	·034906585039886592
Triple =	·052359877559829887
4 ple.. =	·069813170079773183
5 ple.. =	·087266462599716479
6 ple.. =	·104719755119659775
7 ple.. =	·122173047639603070
8 ple.. =	·139626340159546366
9 ple.. =	·157079632679489662

Table II. $\frac{180}{\pi}$ and multiples.

$\frac{180}{\pi}$ =	57·295779513082321
	114·591559026164642
	171·887338539246963
	229·183118052329284
	286·478897565411604
	343·774677078493925
	401·070456591576246
	458·366236104658567
	515·662015617740888

Table III.

TANGENT.	CIRCULAR MEASURE.	DEGREES.
·9	·732815101786506592	41·9872124958166601
·8	·674740942223552663	38·6598082540900906
·7	·610725964389208617	34·992020198558662 1
·6	·540419510270584155	30·9637565320735214
·5	·463647609000806116	26·5650511770779894
·4	·380506377112364886	21·8014094863518118
·3	·291456794477867092	16·6992442339936218
·2	·197395559849880758	11·3099324740202131
·1	·099668652491162027	5·7105931374996425
·09	89758174189950523	5·1427645578842415
8	79829985712237316	4·5739212599008612
7	69886001634642499	4·0041729407093873
6	59928155121207884	3·4336303624505221
5	49958395721942761	2·8624052261117475
4	39978687123290041	2·2906100426385295
3	29991004856877900	1·7183580016554571
2	19997333973150533	1·1457628381751034
1	·009999666686665238	·5729386976834859

* On looking over the tables in this paper, I find one erroneous figure in Table IV., viz. 9 instead of 6 in the common log. of $(1-\cdot000000)^{-1}$, which should be ·0000002605767673 instead of ·0000002605797673.

† I hope I shall be excused for employing this term, which, I believe, has not been made use of before. I have adopted it to avoid the awkward circumlocution of *arc whose tangent is*... I do not see any other term that can be employed, unless it be *inverse* tangent. Having however, in my former paper adopted antilogarithm (which is *sometimes* used) I have preferred antitangent.

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Table III.—*(continued)*.

TANGENT.	CIRCULAR MEASURE.	DEGREES.
·009	·00899975701180911 7	·5156480934199318
8	7999829339886634	·4583564580004315
7	6999885670027949	·4010639059667059
6	5999928001555160	·3437705518714731
5	4999958333958322	·2864765102770744
4	3999978666871464	·2291818957541004
3	2999991000048600	·1718868228800159
2	1999997333339733	·1145914062377860
1	·00099999966666867	·0572957604145006
·0009	899999757000118	51566187638906 4
8	799999829333399	45836613831989 9
7	699999885666700	40107039108342 1
6	599999928000016	34377463582554 2
5	499999958333340	28647887369217 4
4	399999978666669	22918310582923 1
3	299999991000000	17188733382627
2	199999997333333	11459155749827 7
1	·00009999999966667	·0057295779322096
·00009	89999999757000	5156620142254 5
8	79999999829333	4583662351268 1
7	69999999885667	4010704559364 9
6	59999999928000	3437746766659 6
5	49999999958333	2864788973266 8
4	39999999978667	2291831179301 0
3	29999999991000	1718873384876 8
2	19999999997333	1145915590108 9
1	·00000999999999667	·0005729577951117
·000009	8999999999757	515662015603 8
8	7999999999829	458366236094 9
7	6999999999886	401070456585 0
6	5999999999928	343774677074 4
5	4999999999958	286478897563 0
4	3999999999979	229183118051 1
3	2999999999991	171887338538 7
2	1999999999997	114591559026 0
1	1000000000000	·0000572957951 31

Besides these Tables we shall require the value of $\frac{\pi}{2}$, viz.

$$\frac{\pi}{2} = 1{\cdot}570796326794896619.$$

Tables I. and II. do not seem to require any explanation. The last two columns of Table III. contain the antitangents of the numbers of the first column; the second column being expressed in circular measure, and the third in degrees.

2. The antitangents of the numbers in Table III. except the first 7 or 8, may be computed from the series

$$\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \text{ etc. } \ldots\ldots\ldots\ldots (1)$$

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in the same manner as Table IV. of my former paper was from $-\log_e(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 +$ *etc.*, the only difference being that in calculating the third column, $\frac{180}{\pi}$ must be employed instead of the modulus M. This computation need not, therefore, be dwelt upon here.

The series (1) cannot be advantageously used to compute the antitangents of the first 7 or 8 numbers of Table III., for its convergence, when x approaches unity, is slow. Having however, established the antitangents of all the other numbers in the table, we shall be able to effect those of the remainder with comparative ease, for $\tan^{-1}\cdot 2 = \tan^{-1}\cdot 1 + \tan^{-1}\frac{\cdot 1}{1\cdot 02}$;

$\tan^{-1}\cdot 3 = \tan^{-1}.2 + \tan^{-1}\frac{\cdot 1}{1\cdot 06}\ldots\ldots\tan^{-1}\cdot 9 = \tan^{-1}\cdot 8 + \tan^{-1}\frac{\cdot 1}{1\cdot 72}$;

hence if we calculate the values of $\tan^{-1}\frac{\cdot 1}{1\cdot 02}$, $\tan^{-1}\frac{\cdot 1}{1\cdot 06}\ldots\tan^{-1}\frac{\cdot 1}{1\cdot 72}$ (which are expeditiously found by the method about to be explained) we shall readily find, in succession, those of $\tan^{-1}\cdot 2$, $\tan^{-1}\cdot 3\ldots\tan^{-1}\cdot 9$.

Before proceeding further, it is proper to observe that I shall treat of those antitangents only which are comprised between 0 and $\frac{\pi}{2}$; for, since any other case may always be reduced to this, it will not be necessary to consider such as are without these limits.

3. Let it be required to find the antitangent of any fraction $\frac{a}{b}$ less than unity, to p* places of decimals.

Divide a by b, and let the quotient (restricted to one significant figure) be r_1, also let $\tan^{-1}\frac{a}{b} - \tan^{-1} r_1 = \tan^{-1}\frac{a_1}{b_1}$; divide a_1 by b_1, and let the quotient be r_2, and $\tan^{-1}\frac{a_1}{b_1} - \tan^{-1} r_2 = \tan^{-1}\frac{a_2}{b_2}$; repeat this process until the quotient of a_n by b_n begins with $\frac{1}{2}p$ ciphers.

It is evident that we have

$$\tan^{-1}\frac{a}{b} = \tan^{-1} r_1 + \tan^{-1}\frac{a_1}{b_1},$$

$$\tan^{-1}\frac{a_1}{b_1} = \tan^{-1} r_2 + \tan^{-1}\frac{a_2}{b_2},$$

$$\vdots$$

$$\tan^{-1}\frac{a_{n-1}}{b_{n-1}} = \tan^{-1} r_n + \tan^{-1}\frac{a_n}{b_n}.$$

Add these equations, cancel the common terms, and for convenience, put $\frac{a_n}{b_n} = m$.

* I mean p decimals in circular measure; the extent of the approximation in degrees will be about $p - 2$ *decimal* places.

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$$\therefore \tan^{-1}\frac{a}{b} = \tan^{-1} r_1 + \tan^{-1} r_2 \ldots + \tan^{-1} r_n + \tan^{-1} m \ldots (2)$$

Again, $\tan^{-1} m = m - \frac{1}{3}m^3 + \text{etc.}$; now since m begins with $\frac{1}{2}p$ ciphers, m^3 begins with at least p ciphers, *etc.*; hence the correct value of $\tan^{-1} m$ to p places is m, and (2) becomes

$$\tan^{-1}\frac{a}{b} = \tan^{-1} r_1 + \tan^{-1} r_2 \ldots + \tan^{-1} r_n + m \ldots (A)$$

This formula is adapted to circular measure; but if the value of $\tan^{-1}\frac{a}{b}$ be required in degrees, we shall merely have to write $\frac{180}{\pi} \cdot m$ for m, and the formula in this case will take the form

$$\tan^{-1}\frac{a}{b} = \tan^{-1} r_1 + \tan^{-1} r_2 \ldots + \tan^{-1} r_n + \frac{180}{\pi} \cdot m \ldots (B)$$

The values of $\tan^{-1} r_1, \tan^{-1} r_2 \ldots$ in (A) and (B), will be found in the second and third columns of Table III., and $\frac{180}{\pi} \cdot m$, contained in (B), may be easily computed by Table II.

4. Since $\tan^{-1}\frac{a_1}{b_1} = \tan^{-1}\frac{a}{b} - \tan^{-1} r_1 = \tan^{-1}\frac{a - br_1}{b + ar_1}$, we have

$$b_1 = b + ar_1, \quad \text{and} \quad a_1 = a - br_1.$$

Similarly, $b_2 = b_1 + a_1 r_2$ and $a_2 = a_1 - b_1 r_2$,

$$\vdots \qquad\qquad \vdots$$

and $b_n = b_{n-1} + a_{n-1} r_n$ $a_n = a_{n-1} - b_{n-1} r_n$.

Hence the computation of $r_1, r_2 \ldots r_n$ may be thus conducted:

b	a $(r_1$
$+ar_1$	$-br_1$
b_1	a_1 $(r_2$
$+a_1 r_2$	$-b_1 r_2$
b_2	a_2 $(r_3$
$\vdots$	$\vdots$
b_{n-1}	a_{n-1} $(r_n$
$+a_{n-1} r_n$	$-b_{n-1} r_n$
b_n	a_n

It will be observed that the multiplications are here made in cross order: thus a_1 in the second column multiplied by r_2, gives the addend $a_1 r_2$ in the first column; and b_1 in the first column multiplied by r_2, gives the subtrahend $b_1 r_2$ in the second column.

5. Let it be required to find the antitangent of any number a.

If a be less than unity, $\tan^{-1} a$ will be found by the process developed in the last two articles, but if it be greater, calculate $\tan^{-1}\frac{1}{a}$, then will $\frac{\pi}{2} - \tan^{-1}\frac{1}{a}$ be the value of $\tan^{-1} a$.

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We shall now illustrate what has been said by an example or two.

6. Example I.—Required the antitangent of ·548, in circular measure, to 10 or 11 places of decimals.

1	·548 (·5
·274	5
1·274	·048 (·03
144	3822
1·27544	·00978 (·007
6846	892808
1·27550846	·0008|5192 (·0006
51	7|65305076
1·27550897	·0000 86614924 (·000067906166 = m.
||||||||	76530538
	10084386
	8928563
	1155823
	1147958
	7865
	7653
	212
	128
	84
	76
	8
	8

Hence (A) we have

$\tan^{-1}\cdot5 = \cdot463647609001$
$\tan^{-1}\cdot03 = 29991004857$
$\tan^{-1}\cdot007 = 6999885670$
$\tan^{-1}\cdot0006 = 599999928$
$m = 67906166$
$\cdot50130640562 = \tan^{-1}\cdot548.$

Example II.—Required the value of $\tan^{-1} 3$ in degrees.

Here we must first calculate $\tan^{-1}\frac{1}{3}$, (*art.* 5.)

3	1 (·3
·3	·9
3·3	·1 (·03
03	99
3·303	·001 (·0003028 = m.
|||	9909
	91
	66
	25
	26

Hence we have (B and *art.* 5,)

$\tan^{-1}\cdot3 = 16\cdot69924$
$\tan^{-1}\cdot03 = 1\cdot71836$
$\frac{180}{\pi}\cdot m = \begin{cases} 1719 \\ 11 \\ 5 \end{cases}$

$18\cdot43495 = \tan^{-1}\frac{1}{3}.$
90

$71\cdot56505 = 71^\circ\ 33'\ 54'' = \tan^{-1} 3.$

We now proceed to compute the tangent of a given arc or antitangent.

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7. Let it be required to find the tangent of any arc a to p places of decimals, a being less than $\frac{\pi}{4}$.

Take, from the proper column of Table III. $\tan^{-1} r_1$ the nearest arc or antitangent less than a, and let $a - \tan^{-1} r_1 = a_1$; take $\tan^{-1} r_2$ the nearest antitangent less than a_1, and let $a_1 - \tan^{-1} r_2 = a_2$; repeat this operation until the last remainder a_n (or a_n multiplied by $\frac{\pi}{180}$, if a be expressed in degrees) begin with at least $\frac{1}{3}p$ ciphers.

We obviously have
$$\begin{aligned} a &= \tan^{-1} r_1 + a_1, \\ a_1 &= \tan^{-1} r_2 + a_2, \\ &\vdots \\ a_{n-1} &= \tan^{-1} r_n + a_n . \end{aligned}$$

Add these equations together and reject the common terms,

$$\therefore \quad a = \tan^{-1} r_1 + \tan^{-1} r_2 \ldots + \tan^{-1} r_n + a_n \ldots\ldots\ldots (3)$$

Moreover, putting a_n or $\frac{\pi}{180} \cdot a_n$ (as the case requires) $= m$, we have $\tan a_n = m + \frac{1}{3} m^3 + etc.$; and, since m begins with at least $\frac{1}{3}p$ ciphers, the correct value of $\tan a_n$ to p places of decimals evidently $= m$, hence $a_n = \tan^{-1} m$. The equation (3) therefore becomes

$$a = \tan^{-1} r_1 + \tan^{-1} r_2 \ldots + \tan^{-1} r_n + \tan^{-1} m \ldots\ldots\ldots (4)$$

It must be borne in mind that when a is expressed in circular measure, $m = a_n$, but when in degrees, $m = \frac{\pi}{180} \cdot a_n$; in the latter case, the value of m may be easily found by means of the multiples of $\frac{\pi}{180}$ in Table I.

8. Put
$$\begin{aligned} b_n &= 1 - m r_n \quad \text{and} \quad c_n = m + r_n \\ b_{n-1} &= b_n - c_n r_{n-1} \qquad\quad c_{n-1} = c_n + b_n r_{n-1} \\ b_{n-2} &= b_{n-1} - c_{n-1} r_{n-2} \qquad\quad c_{n-2} = c_{n-1} + b_{n-1} r_{n-2} \\ &\vdots \qquad\qquad\qquad\qquad\quad \vdots \\ \text{and,}\quad b_1 &= b_2 - c_2 r_1 \qquad\qquad\quad c_1 = c_2 + b_2 r_1 . \end{aligned}$$

Hence we have, $\tan^{-1} r_n + \tan^{-1} m = \tan^{-1} \frac{m + r_n}{1 - m r_n} = \tan^{-1} \frac{c_n}{b_n}$,

$$\tan^{-1} r_{n-1} + \tan^{-1} \frac{c_n}{b_n} = \tan^{-1} \frac{c_n + b_n r_{n-1}}{b_n - c_n r_{n-1}} = \tan^{-1} \frac{c_{n-1}}{b_{n-1}},$$

$$\vdots$$

$$\text{and,} \quad \tan^{-1} r_1 + \tan \frac{c_2}{b_2} = \tan^{-1} \frac{c_2 + b_2 r_1}{b_2 - c_2 r_1} = \tan^{-1} \frac{c_1}{b_1}.$$

We thus obviously have,

$$\tan^{-1} r_1 + \tan^{-1} r_2 \ldots + \tan^{-1} r_n + \tan^{-1} m = \tan^{-1} \frac{c_1}{b_1};$$

$$\therefore (4), \quad a = \tan^{-1} \frac{c_1}{b_1}, \quad \text{or} \quad \tan a = \frac{c_1}{b_1} \ldots\ldots\ldots\ldots\ldots (C)$$

Hence the value of $\tan a$ is known.

9. The numerical operation required to find the values of b_1 and c_1, may be arranged similarly to that in art. 4, as follows:

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1	$m \quad (r_n$
$-mr_n$	$+r_n$
b_n	$c_n \quad (r_{n-1}$
$-c_n r_{n-1}$	$+b_n r_{n-1}$
b_{n-1}	$c_{n-1} \quad (r_{n-2}$
$\vdots$	$\vdots$
b_2	$c_2 \quad (r_1$
$-c_2 r_1$	$+b_2 r_1$
b_1	c_1

If the value of tan a be required in decimals, as will generally be the case, then the division of c_1 by b_1 must be effected.

10. Let it be required to find tan a when a is greater than $\frac{\pi}{4}$.

Calculate $\frac{c_1}{b_1}$ the value of $\tan\left(\frac{\pi}{2} - a\right)$, as directed above, then will $\tan a = \frac{b_1}{c_1}$, for it is obvious that $\tan a = \frac{1}{\tan\left(\frac{\pi}{2} - a\right)}$.

The following examples will, it is hoped, fully elucidate the processes of articles 7—10.

11. Example I.—Required the tangent of ·261799388 (circular measure) to 8 places of decimals.

·261799388

$\tan^{-1}$ ·2 = 197395560

·064403828

$\tan^{-1}$·06 = 59928155

·004475673

$\tan^{-1}$·004 = 3999979

·000475694(.004

4

·0044756|94(·06

599998|86

·0644755|80(.2

19994591|1

c_1 = 264421491(·26794919

197366888

67054603

59210066

7844537

6907841

936696

888151

48545

39473

9072

8882

190

99

91

89

—

1

·000001903

·999998097

2685|42

·99972955|5

1289511|6

b_1 = ·986834439

||||||||

Hence (C) tan ·261799388 = $\frac{c_1}{b_1}$

= ·26794919.

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Example II.—Required the tangent of 67° 30′.

Here (*art.* 10) we must first find $\tan(90°-67°\ 30') = \tan 22°\ 30'$.

22·5

21·8014095 = $\tan^{-1}\cdot 4$

·6985905

·5729387 = $\tan^{-1}\cdot 01$

·1256518

1145914 = $\tan^{-1}\cdot 002$

a_n = ·0110604

·000174533
17453
1047
7
$= \frac{\pi}{180}\cdot a_n = m.$

·000193|040(·002
2

·0021930|40(·01
99999|96

·01219303|6(·4
39999107|4

c_1 = ·41218411 0(2·41421356.

1
·000000386

·9999996|14
219|30

·99997768|4
487721|4

b_1 = 995100470
824368220

170732250
164873644

5858606
4121841

1736765
1648736

88029
82437

5592
4122

1470
1237

233
206

27
25

Hence (*art.* 10) $\tan 67°\ 30' = \frac{b_1}{c_1}$
$= 2\cdot 41421356.$

12. The values of $\sin^{-1}\frac{a}{b}$, $\cos^{-1}\frac{a}{b}$, $\sec^{-1}\frac{a}{b}$, and $\operatorname{cosec}^{-1}\frac{a}{b}$, and of $\sin a$, $\cos a$, $\sec a$, and $\operatorname{cosec} a$ may also be found by the methods developed in this paper, though the labour will, in general, be much greater than for $\tan^{-1}\frac{a}{b}$ and $\tan a$. For instance, to calculate $\sin^{-1}\frac{a}{b}$, we must compute its equivalent $\tan^{-1}\frac{a}{\sqrt{b^2-a^2}}$; also having found $\tan a = \frac{c_1}{b_1}$ or $\frac{b_1}{c_1}$ (*arts.* 8, 10), we shall have $\sin a = \frac{c_1}{\sqrt{b^2_1+c^2_1}}$ or $\frac{b_1}{\sqrt{b^2_1+c^2_1}}$.

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APPLICATION OF ALGEBRA TO THE MODERN GEOMETRY.

[*Mr. Robert Finlay, Professor of Mathematics and Natural Philosophy, Manchester New College.*]

(Continued from p. 87.)

Relations of involution between the points in which an arbitrary transversal meets a system of surfaces of the first and second degree.

XIX.

If a straight line, drawn from any point P to a fixed point A, meet a given surface of the second degree in Q and Q', it is required to express the ratio $PQ\times PQ' : AQ\times AQ'$ in terms of the coordinates of P referred to any three rectangular axes passing through A.

Let Ax, Ay, Az be the axes, and put $AP = r$, $xAP = l$, $yAP = m$, $zAP = n$, $AQ = \rho'$, $AQ = \rho''$; then

$$\frac{PQ\times PQ'}{AQ\times AQ'} = \frac{(r-\rho')(r-\rho'')}{\rho'\rho''} = \frac{r^2}{\rho'\rho''} - \left(\frac{1}{\rho'} + \frac{1}{\rho''}\right)r + 1 \ldots\ldots(a)$$

Now if the given surface be denoted by

$$u = Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy + Gx + Hy + Kz + 1 = 0,$$

its polar equation will be

$$(A\cos^2 l + B\cos^2 m + C\cos^2 n + D\cos m\cos n + E\cos n\cos l + F\cos l\cos m)\rho^2 + (G\cos l + H\cos m + K\cos n)\rho + 1 = 0;$$

which, for the sake of brevity, we shall denote by

$$M\rho^2 + N\rho + 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(b)$$

Since ρ' and ρ'' are the roots of this equation, we obtain by quadratics

$$\frac{1}{\rho'\rho''} = M, \quad \frac{1}{\rho'} + \frac{1}{\rho''} = -N;$$

hence by substituting these in equation (a) we get

$$\frac{PQ\times PQ'}{AQ\times AQ'} = Mr^2 + Nr + 1 \ldots\ldots\ldots\ldots\ldots\ldots(c)$$

Let x', y', z' be the coordinates of P, so that

$$x' = r\cos l, \quad y' = r\cos m, \quad z' = r\cos n;$$

then (attending to the values of M and N) this equation gives

$$\frac{PQ\times PQ'}{AQ\times AQ'} = u' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where u' denotes the value of u when x', y', z' are put for x, y, z.

Hence it is evident that if P be taken so that the ratio $PQ\times PQ' : AQ\times AQ'$ may be invariable, the locus of P is a surface of the second degree similar and concentric to the given one.

XX.

If PM be drawn from any point P perpendicular to a given plane, and if AN be drawn from a fixed point A perpendicular to the same plane; it is required to express the ratio PM : AN in terms of the coordinates of P referred to any rectangular axes passing through A.

Let $v = \alpha x + \beta y + \gamma z + 1 = 0$ be the equation of the given plane, and $x'y'z'$ the coordinates of P; then

$$PM = \frac{\alpha x' + \beta y' + \gamma z' + 1}{\sqrt{(\alpha^2 + \beta^2 + \gamma^2)}}, \quad AN = \frac{1}{\sqrt{(\alpha^2 + \beta^2 + \gamma^2)}};$$

$$\therefore \frac{PM}{AN} = \alpha x' + \beta y' + \gamma z' + 1 = v' \ldots\ldots\ldots\ldots\ldots(2)$$

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Join AP and produce it to meet the given plane in Q, then by similar triangles PM : AN : : PQ : AQ ; hence

$$\frac{PQ}{AQ} = v' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2')$$

Equations (1), (2), and (2′) are of great importance in translating geometrical relations into algebraic form, and *v. v.* It is evident that the first of them may easily be extended to algebraic surfaces of every degree.

XXI.

Four planes being given, it is required to find the locus of a point P, such that if PM′, PM″ be drawn perpendicular to two opposite planes, and PM_1, PM_2 perpendicular to the other two planes; PM′×PM″ shall be to $PM_1 \times PM_2$ in a given ratio.

Let $v' = 0$, $v'' = 0$, $v_1 = 0$, $v_2 = 0$ be the equations of the planes M, M′, M_1, M_2 respectively, referred to any rectangular axes passing through a fixed point A ; and let AN′, AN″, AN_1, AN_2 be perpendiculars drawn from A to the same planes. Since these perpendiculars are invariable, it is plain that the ratio PM′×PM″ : AN′×AN″ shall be to $PM_1 \times PM_2 : AN_1 \times AN_2$ in a given ratio m to 1. But by equation (2)

$$\frac{PM'}{AN'} = v', \quad \frac{PM''}{AN''} = v'', \quad \frac{PM_1}{AN_1} = v_1, \quad \frac{PM_2}{AN_2} = v_2;$$

hence $v'v'' : v_1v_2 : : m : 1$, or

$$v'v'' = mv_1v_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

and therefore the locus is a surface of the second degree passing through four of the intersections of the four given planes (XII).

This is analogous to the theorem *Ad quatuor lineas* of the ancients, which probably led DESCARTES to his general method of applying algebra to geometry.

Join AP, let it meet the surface again in P′, and the planes M′N′, M″N″, M_1N_1, M_2N_2 in Q, Q′, R, R′ respectively : then by equation (2′),

$$\frac{PQ \times PQ'}{AQ \times AQ'} : \frac{PR \times PR'}{AR \times AR'} : : m : 1 : : \frac{P'Q \times P'Q'}{AQ \times AQ'} : \frac{P'R \times P'R'}{AR \times AR'},$$

$$\therefore \quad PQ \times PQ' \times P'R \times P'R' = P'Q \times P'Q' \times PR \times PR' \ldots . (4)$$

and consequently the six points P, P′, Q, Q′, R, R′ are in involution.

This corresponds to the following theorem of DESARGUES.

If any quadrilateral be described in a conic section, any transversal meets the curve and the sides of the quadrilateral in six points which are in involution.

When the three points R, R′, Q′, and the surface (3) are given, the points P and P′ become known, and the point Q becomes fixed in consequence of the relation (4).

Hence *if four of the intersections of four variable planes move on a given surface of the second degree, and if three of the planes pass through three given points* P, R′ and Q′, *which range upon a straight line, the fourth plane will pass through a fixed point Q, which ranges upon the same straight line.*

This is analogous to a well known theorem which appears to be due to BRIANCHON.—See the Mathematician, Vol. I., p. 289, art. 29.

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XXII.

To find the locus of a point P such that if three perpendiculars PM_1, PM', PM, be drawn from it to three given planes, the rectangle $PM_1 \times PM'$ shall be to the rectangle contained by PM and a given straight line, in a given ratio.

Let $v_1 = 0$, $v' = 0$, $v = 0$ be the equations of the planes M_1, M', M, referred to any rectangular axes passing through a fixed point A. If AN_1, AN', AN be perpendiculars drawn from A to the same planes, it is evident that the ratio $PM_1 \times PM' : AM_1 \times AM'$ is to PM : AN in a given ratio, m to 1 suppose; hence by equation (2)

$$v_1 v' = mv \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and therefore the locus is a surface of the second degree passing through the two straight lines in which the plane v intersects the planes v_1 and v'.

Since m is arbitrary, equation (5) denotes a system of surfaces of the second degree, which are evidently similar; hence *an infinite number of similar surfaces of the second degree may be described so that each of them shall pass through two of the intersections of three given planes.*

Join AP, let it meet the surface again in P', and the planes v_1, v', v, in Q, Q', R respectively; by equation (2')

$$\frac{PQ \times PQ'}{AQ \times AQ'} : \frac{PR}{AR} :: m : 1 :: \frac{P'Q \times P'Q'}{AQ \times AQ'} : \frac{P'R}{AR},$$

$$\therefore \quad PQ \times PQ' \times P'R = P'Q \times P'Q' \times PR \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

and consequently the five points P, P', Q, Q', R, are in involution.

Let the locus (5) pass through two given points P and P', the planes v_1, v' being supposed to be fixed, and the plane v variable; then since the points Q and Q' are determined by the straight line PP', which passes through the two given points, it is evident that the point R becomes fixed (6). Hence *if a surface of the second degree pass through two fixed points, and through the straight lines in which two fixed planes are cut by a movable plane, this plane will turn about a fixed point.*

XXIII.

To find the locus of a point P such that if PM_1, PM', PM, be drawn perpendicular to three given planes, the rectangle $PM_1 \times PM'$ shall be to PM^2 in a given ratio.

Let $v_1 = 0$, $v' = 0$, $v = 0$ be the equations of the planes, referred to any rectangular axes passing through a fixed point A, and let AN_1, AN', AN be perpendicular to the given planes. It is plain that $PM_1 \times PM' : AN_1 \times AN'$ is to $PM^2 : AN^2$ in a given ratio m to 1; or by equation (2), $v_1 v' : v^2 :: m : 1$;

$$\therefore \quad v_1 v' = mv^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

and consequently the locus is a surface of the second degree, touching the planes v_1 and v' along the straight lines in which they are cut by the plane v (XIV).

Join AP, let it meet the surface again in P', and the planes v_1, v', v, in Q, Q' and R respectively. By equation (2')

$$\frac{PQ \times PQ'}{AQ \times AQ'} : \frac{PR^2}{AR^2} :: m : 1 :: \frac{P'Q \times P'Q'}{AQ \times AQ'} : \frac{P'R^2}{AR^2};$$

$$\therefore \quad PQ \times PQ' \times P'R^2 = P'Q \times P'Q' \times PR^2 \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

therefore the points P, P', Q, Q', R are in involution, R being a double point.

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XXIV.

To find the locus of a point P, such that if a straight line be drawn from it to any given point A, meeting a given surface of the second degree in Q, Q', and two given planes in R, R'; the ratio $PQ\times PQ' : AQ\times AQ'$ shall be to $\frac{PR\times PR'}{AR\times AR'}$ in a given ratio.

The point A being taken as the origin of rectangular axes, let $u = 0$ denote the equation of the given surface, and $v = 0$, $v' = 0$ the equations of the planes. By equations (1) and (2') we obtain

$$\frac{PQ\times PQ'}{AQ\times AQ'} = u, \quad \frac{PR}{AR} = v, \quad \frac{PR'}{AR'} = v';$$

$$\therefore\ u = mvv' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

and consequently the locus is a surface of the second degree, passing through the curves (real or imaginary) in which the planes intersect the given surface (I).

If P' be the other point in which AP meets the locus, we shall have (9)

$$\frac{PQ\times PQ'}{AQ\times AQ'} = m\times\frac{PR\times PR'}{AR\times AR'}, \quad \frac{P'Q\times P'Q'}{AQ\times AQ'} = m\times\frac{P'R\times P'R'}{AR\times AR'},$$

$$\therefore\ PQ\times PQ'\times P'R\times P'R' = P'Q\times P'Q'\times PR\times PR' \ldots\ldots (10)$$

Hence *if two surfaces of the second degree intersect each other along two plane curves, and if any transversal meet the surfaces and the planes of intersection in six points, these points are in involution.*

This is analogous to the following theorem *in plano*.

If two conic sections be described about a quadrilateral, and if any transversal cut the curve in four points, and two opposite sides of the quadrilateral in two other points; these six points are in involution.—STURM, Annal. des Math., tom. XVII.

When two points P, P' of the locus are given, Q and Q' are fixed points; consequently if one of the planes $v' = 0$, be fixed and the other moveable, the five points P, P', Q, Q', R', are fixed, and therefore R is also fixed (10): hence, in this case, the moveable plane $v = 0$ will turn about a fixed point R. This result was already obtained (II) by a different process of reasoning.

When the planes $v = 0$ and $v' = 0$ are fixed, if one point P' of the locus be given, the locus may be constructed by points. For if any straight line be drawn through P' meeting the given surface in Q and Q' and the given planes in R and R', the five points Q, Q', R, R', P' are fixed, and the position of P may be calculated from (10).

XXV.

To find the locus of a point P such that if a straight line be drawn from it to a given point A, meeting a given surface of the second degree in Q and Q', and a given plane in R; the ratio $PQ\times PQ' : AQ\times AQ'$ shall be to PR : AR in a given ratio m to 1.

The axes of coordinates being taken as in the last proposition, let $u = 0$ be the equation of the given surface, $v = 0$ the equation of the plane; then by (1) and (2')

$$u : v :: m : 1, \text{ or}$$

$$u = mv \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

therefore (IX) the locus is a surface of the second degree intersecting the given surface on the single plane $v = 0$.

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If AP meet the locus again in P′, we shall have

$$\frac{PQ\times PQ'}{AQ\times AQ'}=m\times\frac{PR}{AR},\quad\frac{P'Q\times P'Q'}{AQ\times AQ'}=m\times\frac{P'R}{AR},$$

$$\therefore\ PQ\times PQ'\times P'R=P'Q\times P'Q'\times PR\ldots\ldots\ldots\ldots(12)$$

and therefore the five points P, P′, Q, Q′, R are in involution.

Hence *if any transversal meet two similar surfaces of the second degree, and the plane in which they intersect, in five points, these points are in involution.*

XXVI.

To find the locus of a point P such that if a straight line be drawn from it to a given point A, meeting a given surface of the second degree in Q, Q′ and a given plane in R; then $PQ\times PQ' : AQ\times AQ'$ shall be to $PR^2 : AR^2$ in a given ratio m to 1.

The axes of coordinates being assumed as before, let $u=0$ be the equation of the given surface, $v=0$ that of the given plane; then by (1) and (2′) we obtain

$$u=mv^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

and therefore the locus is a surface of the second degree touching the given surface along the curve in which it is cut by the given plane (v).

If AP meet the locus again in P′, we shall have

$$\frac{PQ\times PQ'}{AQ\times AQ'}=m\frac{PR^2}{AR^2},\quad\frac{P'Q\times P'Q'}{AQ\times AQ'}=m\frac{P'R^2}{AR^2},$$

$$\therefore\ PQ\times PQ'\times P'R^2=P'Q\times P'Q'\times PR^2\ldots\ldots\ldots(14)$$

and consequently the points P, P′, Q, Q′, R are in involution, and R is a *double* point.—*Mathematician*, Vol. I., p. 245.

Hence *if a surface of the second degree touch a given surface of the second degree along a plane curve, and if any transversal cut the two surfaces and the plane of contact in five points; these points are in involution, and the point in which the transversal cuts the plane of contact is double.*

XXVII.

To find the locus of a point P, such that if a straight line be drawn through it and a given point A, meeting two given surfaces of the second degree in Q, Q′ and R, R′ respectively; $PQ\times PQ' : AQ\times AQ'$ shall be to $PR\times PR' : AR\times AR'$ in a given ratio.

The axes being taken as before, let $u=0$ and $u'=0$ be the equations of the given surfaces; then by equation (1)

$$u=mu'\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

therefore the locus is a surface of the second degree passing through the intersection of the two given surfaces.

If P′ be the other point in which AP meets the locus, we shall have

$$\frac{PQ\times PQ'}{AQ\times AQ'}=m\frac{PR\times PR'}{AR\times AR'},\quad\frac{P'Q\times P'Q'}{AQ\times AQ'}=m\frac{P'R\times P'R'}{AR\times AR'},$$

$$\therefore\ PQ\times PQ'\times P'R\times P'R'=P'Q\times P'Q'\times PR\times PR'\ldots\ldots(16)$$

Hence *if a surface of the second degree pass through the intersection of two given surfaces of the second degree, and if any transversal cut the three surfaces in six points; these points are in involution.*

(To be continued.)

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ELEMENTARY INVESTIGATION OF THE FORMULA FOR COMPUTING THE EQUATION OF EQUAL ALTITUDES.

[*Mr. John Riddle, Greenwich.*]

Let p and p_1 = the polar distances, p being the greater,
h and h_1 = the hour angles, then h_1 is the greater,
z and z = the equal zenith distances,
l = colatitude, and L = latitude.

Also let P = polar distance at noon $= \frac{1}{2}(p+p_1)$
C = ½ change of declination $= \frac{1}{2}(p-p_1)$
I = ½ elapsed time $= \frac{1}{2}(h_1+h)$
E = equation of equal altitudes $= \frac{1}{2}(h_1-h)$

C and E being very small are taken for their sines.

Then $\sin p.\ \sin l.\ \cos h = \cos z - \cos p.\ \cos l,$
and $\sin p_1.\ \sin l.\ \cos h_1 = \cos z - \cos p_1.\cos l.$

Subtracting and dividing by $\sin l$, we obtain

$$\sin p.\ \cos h - \sin p_1\ \cos h_1 = \cot l.\ (\cos p_1 - \cos p).$$

Now the second side of this equation

$$= \cot l.\ 2\sin P.\ \sin C$$
$$= 2C.\tan L \sin P \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

And the first side

$$= \tfrac{1}{2}\{\sin(p+h)+\sin(p-h)-\sin(p_1+h_1)-\sin(p_1-h_1)\}$$

And taking the alternate terms together, this expression

$$= \cos\tfrac{1}{2}\{(p+p_1)+(h_1+h)\}\sin\tfrac{1}{2}\{(p-p_1)-(h_1-h)\}$$
$$+\cos\tfrac{1}{2}\{(p+p_1)-(h_1+h)\}\sin\tfrac{1}{2}\{(p-p_1)+(h_1-h)\}$$
$$= \cos(P+I)\sin(C-E)+\cos(P-I)\sin(C+E)$$
$$= \cos(P+I)(C-E)+\cos(P-I)(C+E)$$
$$= C.\{\cos(P+I)+\cos(P-I)\}+E.\{\cos(P-I)-\cos(P+I)\}$$
$$= 2C.\cos P.\ \cos I+2E.\ \sin P.\ \sin I \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Equating (1) and (2) and dividing by 2,

$$C.\ \cos P.\ \cos I.+E.\ \sin P.\ \sin I = C.\ \tan L.\ \sin P\ ;$$
$$\therefore\quad E \sin P.\ \sin I = C.\ \tan L.\ \sin P - C.\ \cos P.\ \cos I\ ;$$
$$\therefore\quad E = \frac{C.\tan L}{\sin I} - \frac{C.\cot P}{\tan I} = C.\tan L.\ \mathrm{cosec}\, I - C.\cot P.\cot I.$$

Now let d = seconds in the change of declination from preceding to following noon.

Then $\dfrac{I.d}{48}$ = change in I hours = C in seconds of arc;

$$\therefore\quad \frac{2.\,I.\,d}{30\times 48} = \frac{2.\,I.\,d}{1440} = C \text{ in seconds of time.}$$

And in Tab. XVI, Baily's Tables, or Tab. XVIII, Riddle's Navigation,

$$\text{Log A} = \log \frac{2.I^{\text{hrs}}}{1440}.\ \mathrm{cosec}\, I^\circ, \text{ and } \log B = \log \frac{2.I^{\text{hrs}}}{1440}.\ \cot I^\circ\ ;$$
$$\therefore\quad E = A.\ d.\ \tan l - B.\ d.\ \cot P.$$

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ON THE VERTICAL FORCE MAGNETOMETER.

[*Mr. Hearn.*]

The following little essay was composed some time since at the request of a scientific friend, and may perhaps interest some of the readers of the Mathematician.

1. A magnetic needle suspended on its centre of gravity is constrained to move in a vertical plane, making an angle I with the magnetic meridian; θ is the dip in the magnetic meridian, and ϕ the dip in the vertical plane in which we suppose the motion to be.

Refer the system to rectangular axes, x being at right angles to the plane of the magnetic meridian, y horizontal and in that plane, and z vertical. Then the vertical and horizontal components are

$$F = M \sin\theta, \quad H = M \cos\theta,$$

and the statical equation of virtual velocities is

$$F\delta z + H\delta y = 0.$$

But $y = a\cos\phi\cos I$, $z = a\sin\phi$, where a is the distance of any particle of the needle from its centre;

$$\therefore \quad \delta y = -a\sin\phi\cos I\delta\phi, \quad \delta z = a\cos\phi\delta\phi;$$

$$\therefore \quad F\cos\phi = H\sin\phi\cos I,$$

or
$$\sin\theta\cos\phi = \cos\theta\sin\phi\cos I,$$

$$\therefore \quad \cot\phi = \cot\theta\cos I \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

which determines ϕ, and thence the position of equilibrium.

The equation $F\cos\phi = H\sin\phi\cos I$

gives
$$F = M\cos\theta\tan\phi\cos I,$$

which gives F when M, *etc.* are known; and by logarithmic differentiation,

$$\frac{\Delta F}{F} = \frac{\Delta M}{M} - \frac{\sin\theta}{\cos\theta}.\Delta\theta + \frac{\sec^2\phi}{\tan\phi}\Delta\phi,$$

a formula different from but equivalent to that of Biot.

Biot's formula is easily found from the same source, thus:

$$F = M\cos\theta\tan\phi\cos I,$$

and
$$\sin\theta\cos\phi = \cos\theta\sin\phi\cos I;$$

$$\therefore \quad \cos^2\phi - \cos^2\phi\cos^2\theta = \cos^2\theta\cos^2 I(1 - \cos^2\phi);$$

$$\therefore \quad \cos^2\theta\cos^2 I = \cos^2\phi\{1 - \cos^2\theta\sin^2 I\},$$

or,
$$\frac{\cos\theta\cos I}{\cos\phi} = \sqrt{(1 - \cos^2\theta\sin^2 I)};$$

$$\therefore \quad F = M\sin\phi\sqrt{(1 - \cos^2\theta\sin^2 I)},$$

and taking the log differentiation

$$\frac{\Delta F}{F} = \frac{\Delta M}{M} + \cot\phi\Delta\phi + \tfrac{1}{2}.\frac{\sin 2\theta\sin^2 I}{1 - \cos^2\theta\sin^2 I}.\Delta\theta,$$

the formula of Biot.

2. When the needle is counterpoised by means of a brass weight so as to remain in equilibrium in a given position, we have only to substitute in the statical equation, $F - \mu$ for F, where μ is a constant moment;

$$\therefore \quad F = \mu + M\cos\theta\tan\phi\cos I.$$

When the position is accurately horizontal, $\phi = 0$; therefore $F = \mu$, or $M\sin\theta = \mu$, and therefore when the needle deviates from horizontality by a very small angle $\Delta\phi$, we have nearly

$$F = \mu + \mu \cot\theta \cos I \Delta\phi,$$

and $$\frac{F-\mu}{\mu} = \frac{\Delta F}{F} = \cot\theta \cos I \Delta\phi.$$

To make this formula available, it appears that the difficulty is to estimate I, and this may be accomplished as follows. Let the needle be slightly moved in the vertical plane; then the vertical component called into action will be $F - \mu$, or $\mu \cot\theta \cos I \Delta\phi$; or since $\Delta\phi$ is the arc described, the *pendulum* force will be $\mu \cot\theta \cos I$. Then let T be the time of vibration. Afterwards let the needle be so adjusted as to vibrate in a horizontal plane under the action only of the horizontal component $M \cos\theta$, and let T′ be the time of vibration, and we will have by the theory of pendulums

$$\frac{T'^2}{T^2} = \frac{\mu \cot\theta \cos I}{\mu \cot\theta} = \cos I;$$

hence $$\frac{\Delta F}{F} = \frac{T'^2}{T^2} \cot\theta \Delta\phi.$$

The utility of this formula is thus explained. If we could place the needle so as to vibrate exactly in the magnetic meridian so much the better; in that case $T' = T$, and we have simply $\frac{\Delta F}{F} = \cot\theta \Delta\phi$; but since we can never be sure of this, the deviation may be estimated from the ratio $\frac{T'^2}{T^2}$.

SOLUTIONS OF MATHEMATICAL EXERCISES.

LXI. *Mr. Thomas Weddle, Newcastle.*

1. Let ABC, A_1BC, AB_2C, and ABC_3 be an associated system of spherical triangles; $OO_1O_2O_3$ the centres of the inscribed, and $QQ_1Q_2Q_3$ those of the circumscribed circles. Let QK, Q_1K_1, Q_2K_2, and Q_3K_3 be perpendiculars upon BC; QL, Q_1L_1, Q_2L_2, and Q_3L_3 perpendiculars upon AC, and QM, Q_1M_1, Q_2M_2, and Q_3M_3 perpendiculars upon AB. Then will

$$\begin{aligned} &\cot AO_2 \cot AO_3 - \cot AO \cot AO_1 = 1 \\ &\cot BO_1 \cot BO_3 - \cot BO \cot BO_2 = 1 \\ &\cot CO_1 \cot CO_2 - \cot CO \cot CO_3 = 1 \end{aligned}$$

and,
$$\begin{aligned} &\tan Q_2K_2 \tan Q_3K_3 + \tan QK \tan Q_1K_1 = 1 \\ &\tan Q_1L_1 \tan Q_3L_3 + \tan QL \tan Q_2L_2 = 1 \\ &\tan Q_1M_1 \tan Q_2M_2 + \tan QM \tan Q_3M_3 = 1. \end{aligned}$$

[First Solution. *Mr. John Laws, Newcastle-on-Tyne.*]

Let the radii of the circles inscribed in the associated system of spherical triangles be denoted by r, r_1, r_2, r_3; then

$$\sin AO = \frac{\sin r}{\sin \frac{1}{2}A}, \qquad \cos AO = \cos r \cos(s-a),$$

$$\sin AO_1 = \frac{\sin r_1}{\sin \frac{1}{2}A}, \qquad \cos AO_1 = \cos r_1 \cos s,$$

$$\sin AO_2 = \frac{\sin r_2}{\cos \frac{1}{2}A}, \qquad \cos AO_2 = \cos r_2 \cos(s-c),$$

$$\sin AO_3 = \frac{\sin r_3}{\cos \frac{1}{2}A}, \qquad \cos AO_3 = \cos r_3 \cos(s-b);$$

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$$\therefore \quad \cot AO_2 \cot AO_3 = \cos^2\tfrac{1}{2}A \cot r_2 \cot r_3 \cos(s-b)\cos(s-c),$$
$$\cot AO \cot AO_1 = \sin^2\tfrac{1}{2}A \cot r \cot r_1 \cos s \cos(s-a).$$

But $\cos^2\tfrac{1}{2}A = \dfrac{\sin s \sin(s-a)}{\sin b \sin c}$, $\sin^2\tfrac{1}{2}A = \dfrac{\sin(s-b)\sin(s-c)}{\sin b \sin c}$,

$$\cot r_2 \cot r_3 = \frac{1}{\sin s \sin(s-a)}, \quad \cot r \cot r_1 = \frac{1}{\sin(s-b)\sin(s-c)};$$

$$\therefore \cot AO_2 \cot AO_3 - \cot AO \cot AO_1 = \frac{\cos(s-b)\cos(s-c) - \cos s \cos(s-a)}{\sin b \sin c} = 1.$$

Similarly, $\cot BO_1 \cot BO_3 - \cot BO \cot BO_2 = 1$,
$\cot CO_1 \cot CO_2 - \cot CO \cot CO_3 = 1$.

Again,

$$\tan QK = \frac{\sin\tfrac{1}{2}a}{\cot(S-A)}, \quad \tan Q_1K_1 = -\frac{\sin\tfrac{1}{2}a}{\cot S},$$
$$\tan Q_2K_2 = \frac{\cos\tfrac{1}{2}a}{\cot(S-C)}, \quad \tan Q_3K_3 = \frac{\cos\tfrac{1}{2}a}{\cot(S-B)};$$

$$\therefore \tan QK \tan Q_1K_1 = -\frac{\sin^2\tfrac{1}{2}a}{\cot S \cot(S-A)},$$

$$\tan Q_2K_2 \tan Q_3K_3 = \frac{\cos^2\tfrac{1}{2}a}{\cot(S-B)\cot(S-C)}.$$

But $\sin^2\tfrac{1}{2}a = -\dfrac{\cos S \cos(S-A)}{\sin B \sin C}$, $\cos^2\tfrac{1}{2}a = \dfrac{\cos(S-B)\cos(S-C)}{\sin B \sin C}$;

$$\therefore \quad \tan Q_2K_2 \tan Q_3K_3 + \tan QK \tan Q_1K_1 = \frac{\sin S \sin(S-A) + \sin(S-B)\sin(S-C)}{\sin B \sin C} = 1.$$

Similarly, $\tan Q_1L_1 \tan Q_3L_3 + \tan QL \tan Q_2L_2 = 1$,
and, $\tan Q_1M_1 \tan Q_2M_2 + \tan QM \tan Q_3M_3 = 1$.

[SECOND SOLUTION. *Messrs. Samuel Bills, Hawton; J. W. Elliott, Greatham; G. W. Hearn, R.M. College, Sandhurst; Nicholas Smyth, Galway; and Thomas Weddle, the Proposer.*]

Adopting the usual notation we shall have from trigonometry and well known properties of the associated triangles,

$$\cot AO = \cot(s-a)\cos\tfrac{1}{2}A, \quad \cot AO_1 = \cot s \cos\tfrac{1}{2}A;$$

whence $\cot AO . \cot AO_1 = \cot s . \cot(s-a) . \cos^2\tfrac{1}{2}A = \dfrac{\cos s \cos(s-a)}{\sin b \sin c}$.

Similarly, $\cot AO_2 \cot AO_3 = \dfrac{\cos(s-b)\cos(s-c)}{\sin b \sin c}$.

Hence,

$$\cot AO_2 \cot AO_3 - \cot AO \cot AO_1 = \frac{\cos(s-b)\cos(s-c) - \cos s \cos(s-a)}{\sin b \sin c}.$$

But $\cos(s-b)\cos(s-c) = \tfrac{1}{2}\cos a + \tfrac{1}{2}\cos(b-c)$,
and $\cos s \cos(s-a) = \tfrac{1}{2}\cos a + \tfrac{1}{2}\cos(b+c)$;
$\therefore \cos(s-b)\cos(s-c) - \cos s \cos(s-a) = \tfrac{1}{2}\cos(b-c) - \tfrac{1}{2}\cos(b+c) = \sin b \sin c$.

The above expression therefore becomes,

$$\cot AO_2 \cot AO_3 - \cot AO \cot AO_1 = 1.$$

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Again, using Mr. Weddle's notation,

$\tan QK = \tan(S-A)\sin\frac{1}{2}a$, $\tan Q_1K_1 = -\tan S \sin\frac{1}{2}a$;

$\therefore \tan QK \tan Q_1K_1 = -\tan S \tan(S-A)\sin^2\frac{1}{2}A = \dfrac{\sin S \sin(S-A)}{\sin B \sin C}$.

In like manner, $\tan Q_2K_2 \tan Q_3K_3 = \dfrac{\sin(S-B)\sin(S-C)}{\sin B \sin C}$.

Also, $\sin(S-B)\sin(S-C) = \frac{1}{2}\cos(B-C) - \frac{1}{2}\cos A$,

and, $\sin S \sin(S-A) = \frac{1}{2}\cos A - \frac{1}{2}\cos(B+C)$.

Hence,

$\sin(S-B)\sin(S-C) + \sin S \sin(S-A) = \frac{1}{2}\cos(B-C) - \frac{1}{2}\cos(B+C) = \sin B \sin C$.

$\therefore \tan Q_2K_2 \tan Q_3K_3 + \tan QK \tan Q_1K_1 = \dfrac{\sin B \sin C}{\sin B \sin C} = 1$.

In a similar manner the other properties are established.

We have also received an excellent solution from Mr. R. Harley, Northwich, Cheshire, which is reluctantly omitted for want of space.

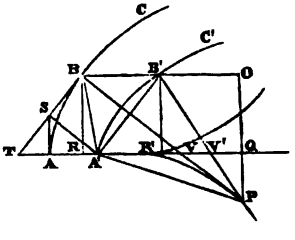
LXII. *Mr. James Reid, Prior's Salford.*

If two parabolas be described on the same axis so that the vertex of the second may be the focus of the first, and its parameter half that of the other; then if from the points in which normals of the smaller parabola meet the evolute of the larger, as centres, and the respective normals as radii, circles be described, these will all pass through a certain fixed point.

Remark.—For "respective normals as radii," read, parts of the normals intercepted between the evolute and the smaller parabola as radii.

[First Solution. *By Geometricus.*]

Let ABC, A′B′C′ be the two parabolas, the vertex A′ of the latter coinciding with the focus of the former. Take any point B in the parabola ABC, and draw BB′ parallel to the axis AA′ intersecting the parabola A′B′C′ in B′: draw also the normals BP, B′P at the points B, B′ meeting the axis in V, V′, and each other in P. Draw the tangent BT, and the ordinates BR, B′R′, and let PQO be drawn perpendicular to the axis intersecting it in Q and meeting BB′ produced in O. Lastly, draw A′S perpendicular to BT, and join A′B, A′B′ and A′P.



Then the subnormal $RV = 2AA'$, and the parameter of ABC being double that of A′B′C′, the subnormal $R'V' = AA'$; hence $RV = 2R'V'$.

Now $BR^2 = 4AA'.AR$, and $B'R'^2 = 2AA'.A'R'$; but $BR^2 = B'R'^2$; therefore $A'R' = 2AR$; consequently

$BB' = RR' = RA' + A'R' = RA' + 2AR = AR + AA' = A'T = A'B = A'V$,

since TBV is a right angle; hence also the chord A′B′ is parallel to the tangent BT and perpendicular to the normal BP. But the normal BP makes equal angles with the lines A′B and BB′, and since it has been shewn that A′B = BB′; therefore it is evident that PA′ = PB′, and a circle described with centre P and radius PB′ will pass through A′, the vertex of the parabola A′B′C′.

Since A′S intersects the tangent BT in the tangent AS at the vertex, we have $A'S^2 = AA'.A'T$ and $A'B^2 = A'T^2$; therefore $A'B^2 : A'S^2 :: A'T$ or

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A'B : AA'; but BP : BV :: BO or RQ : RV :: 2RR' : 2AA' :: A'T or A'B : AA'; therefore A'B² : A'S² :: BP : BV, or A'S² : A'B² :: BV or 2A'S : BP :: A'S : ½BP; consequently A'S : A'B :: A'B : ½BP, and therefore (Wallace's Conic Sections, Prop. IV., p. 152) the line BP is the radius of the circle of curvature at the point B, and P is a point in the evolute of the parabola ABC; but it has been already shewn that PB' = PA'; hence the truth of this curious property is established.

Further, since RR' = A'V; therefore RA' = R'V; but R'V' = AA', hence also VV' = AR.

Now the line BB'O being parallel to AVV', we have by similar triangles,

$$BB' : VV' :: B'O : V'Q, \text{ or } AR + AA' : AR :: R'Q : V'Q;$$

hence, by division, AA' : AR :: R'V' or AA' : V'Q; therefore V'Q = AR = VV'; and consequently by a preceding analogy BB' = B'O, and therefore we have

$$AQ = AR + 2RR' = 3AR + 2AA' \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Again, $BB' : VV' :: PO : PQ$, or $AR + AA' : AR :: PO : PQ$;

hence, by division, $AA' : AR :: BR : PQ$, consequently,

$$AA'.PQ = AR.BR \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The equations (1) and (2) evidently furnish the values of the coordinates of the centre of the circle of curvature at any point in the parabola.—(*Mathematician*, Vol. I. p. 210.)

[SECOND SOLUTION. *Mr. Thomas Weddle.*]

If k be the ordinate of that point in the larger parabola, of which the point (α, β) in the evolute is the centre of curvature, we shall have (*Young's Diff. Cal.*, p. 177),

$$\alpha = \frac{3k^2}{4m} + 2m, \text{ and } \beta = -\frac{k^3}{4m^2} \ldots\ldots\ldots\ldots (1)$$

where $4m$ is the parameter. The equation of the smaller parabola (the vertex of the larger one being the origin) is

$$y^2 = 2m(x - m) \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Also if the normal at the point (x, y) pass through (α, β), its equation will give

$$m(\beta - y) + y(\alpha - x) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Eliminate α, β and x from (3) by means of (1, 2), then

$$k^3 - 3yk^2 + 2y^3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The values of k derived from this equation are, y and $(1 \pm \sqrt{3})y$; the latter must be rejected, as they do not give the property enunciated;

$$\therefore k = y \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

From (1, 2, 5) we readily get $(x - \alpha)^2 + (y - \beta)^2 = (m - \alpha)^2 + \beta^2$, that is, the normal mentioned in the question $= \sqrt{\{(m - \alpha)^2 + \beta^2\}} \ldots\ldots\ldots (6)$

Hence, X, Y being the current coordinates, the equation of the circle is

$$(X - \alpha)^2 + (Y - \beta)^2 = (m - \alpha)^2 + \beta^2 \ldots\ldots\ldots (7)$$

By mere inspection we see that this circle, whatever α and β may be, always passes through the point $(m, 0)$ on the axis of x.

Cor. It appears from (5), that if a point be taken on each parabola so as to have equal ordinates, the normals at these points will intersect upon the evolute of the larger parabola.

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[THIRD SOLUTION. *Mr. J. Reid, the Proposer.*]

Let the equation to the two parabolas be

$$y^2 = 4mx \ldots\ldots\ldots\

For the parabola (2) this becomes

$$y^2 = \frac{2}{27m'}(x - 4m')^3 = \frac{4}{27m}(x - 2m)^3.$$

The locus, therefore, of the centres of the circles which pass through the vertex of the parabola (2) and have a common tangent with it is identical with the evolute of the parabola (1), which proves the proposition.

Good solutions were received from Messrs. John Laws; Nicholas Smyth; and R. H. Wright.

LXIII. *From the Putney College Examination Papers, July*, 1845.

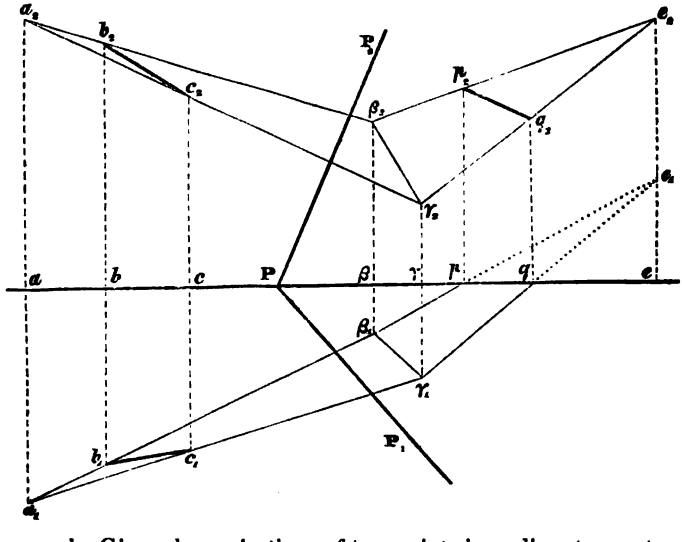
The ground plan and picture being taken as planes of projection, there are given two points, and a finite line by means of their projections, and a plane by means of its traces. If light emanate from one of the given points, and the given line enlightened by it cast a shadow on the given plane, it is required:—

(1) To construct the projections of the shadow on the planes of projection.

(2) To construct the perspective of that shadow, the eye being in the second given point.

[SOLUTION. *Mr. C. H. Brown, Student, C. C. E.*]

The solution is readily effected by means of the two following elementary problems, which are given in all works on *Descriptive Geometry*, and which it is sufficient to merely quote as *Lemmas*. See, for instance, *Lefebure de Fourcy, Traité de Géométrie Descriptive*, probs. i. and iii. pp. 44 and 49.



Lemma 1. Given the projections of two points in a line, to construct the traces of that line.

Lemma 2. Given the projections of two points in a line, and the traces of a plane, to construct the projections of the intersection of the line and plane.

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Take A as the luminous point, BC the given line, and E the place of the eye, respectively denoted by their horizontal projections a_1, b_1c_1, e_1, and their vertical projections a_2, b_2c_2, e_2; and let P_1PP_2 be the given plane denoted by its traces, PP_1 and PP_2.

Now the shadow of the extremity B of the given line is the intersection of the straight line AB through the luminous point A and the point B. Let this intersection be found (*Lemma* 2) and its projections on the two planes be $\beta_1\beta_2$. Similarly let $\gamma_1\gamma_2$ the projections of the intersection of AC with the plane P_1PP_2 be found. Then $\beta_1\gamma_1$ will be the horizontal projection of the shadow, and $\beta_2\gamma_2$ the vertical. The projections of the shadow are therefore found.

To find the perspective of that shadow, we have only to find where lines drawn from its extremities to the eye pierce the vertical plane of projection, which in the problem is the picture plane. But we have the horizontal projections $e_1\beta_1$ and $e_1\gamma_1$ and the vertical projections $e_2\beta_2$ and $e_2\gamma_2$; and hence (*Lemma* 1) we find the vertical traces p_2q_2 of those lines. The line p_2q_2 is the perspective required.

Note. In this sketch the eye is taken on the opposite side of the picture plane from the other data, as in *perspective* should be the case. Divested, however, of the technical application of the problem, the data may have any position whatever: but in such case, the line which represents the perspective might not fall within the limits of the picture.

Good solutions were sent by Messrs. Godfray and Hearn.

LXIV. *Pen-and-Ink.*

When Captain Cook arrived at the Antipodes of London, he remarked in his journal that he was "as far from his friends as he could be;" was he so?

[First Solution. *Mr. G. W. Hearn.*]

This exercise may be geometrically expressed thus: To find the greatest chord which can be drawn in an ellipse from a given point P in the circumference. Let m and n be the distances of the given point P from the minor and major axes, so that

$$\frac{m^2}{a^2} + \frac{n^2}{b^2} = 1,$$

Let r be the length of the chord, and θ the angle it makes with the axis major; then

$$\text{or,} \qquad r\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) - 2\left(\frac{m\cos\theta}{a^2} + \frac{n\sin\theta}{b^2}\right) = 0 \ldots\ldots\ldots(1)$$

Differentiating with respect to θ, and making $\frac{dr}{d\theta} = 0$, we have

$$r\left(-\frac{\sin\theta\cos\theta}{a^2} + \frac{\cos\theta\sin\theta}{b^2}\right) - \left(-\frac{m\sin\theta}{a^2} + \frac{n\cos\theta}{b^2}\right) = 0,$$

$$\text{or,} \qquad r.\frac{a^2-b^2}{a^2b^2}\sin\theta\cos\theta = \frac{n\cos\theta}{b^2} - \frac{m\sin\theta}{a^2}.$$

Eliminating r between this and equation (1) we have

$$ma^2b^2\tan^3\theta + na^2(a^2-2b^2)\tan^2\theta + mb^2(2a^2-b^2)\tan\theta = na^2b^2.$$

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If $x = \tan\theta$, and we make $\frac{n}{m} = \tan\lambda$, this equation, in an ellipse of small eccentricity, becomes

$$x^3 - (1-e^2)\tan\lambda . x^2 + (1+e^2)x - \tan\lambda = 0.$$

Let $x = \tan\lambda - u$, where u is very small; then we find on neglecting u^2, *etc.*

$$u = \frac{e^2\tan\lambda \sec^2\lambda}{1+\tan^2\lambda} = e^2\tan\lambda \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Let therefore λ' be the geographical latitude required, then since the chord is a normal at the point farthest from P,

$$\tan\lambda' = (1-e^2)\tan\theta,$$

and $$\tan\theta = (1-e^2)\tan\lambda, \text{ by } (2)$$

$$\therefore \quad \tan\lambda' = (1-e^2)^2\tan\lambda = (1-4e)\tan\lambda,$$

where e, the compression in the case of the earth, is about $\frac{1}{300}$. Taking London at 51° 30′ N. latitude, we find from this formula, $\lambda' = 51° 7' 33''$, S. latitude; so that when Captain Cook was at the antipodes he was not mathematically speaking farthest from his friends (supposed to be in London), but required to sail about $22\frac{1}{2}$ miles farther north before he could strictly be at the greatest distance from them.

[Second Solution. *Mr. Thomas Weddle.*]

The place (P) at the greatest distance from London (L) is on the opposite meridian, for it is plain that the line drawn from London to that place must be perpendicular to the tangent plane.

Let the major axis $= 2a$, the excentricity $= e$, and the minor axis, or $2a\sqrt{1-e^2} = 2b$. Denote the latitude of London by λ and that of P by $\lambda - \delta$; since the earth is nearly a sphere it is clear that δ as well as e is very small. Denote the points L and P by (x_1, y_1) and $(-x_2, -y_2)$. Moreover, λ being the angle which the normal at L makes with the major axis, we have $\tan\lambda = \frac{a^2y_1}{b^2x_1}$; hence, since $a^2y_1^2 + b^2x_1^2 = a^2b^2$, and $b^2 = a^2(1-e^2)$;

$$\therefore \quad x_1 = \frac{a\cos\lambda}{\sqrt{1-e^2\sin^2\lambda}} = m\cos\lambda, \quad y_1 = \frac{a(1-e^2)\sin\lambda}{\sqrt{1-e^2\sin^2\lambda}} = m(1-e^2)\sin\lambda \ldots (1)$$

where $m = a(1-e^2\sin^2\lambda)^{-\frac{1}{2}}$. In like manner recollecting that $1-e^2\sin^2(\lambda-\delta) = 1-e^2\sin^2\lambda$ very nearly, (since e and δ are small), we have

$$\left.\begin{aligned} x_2 &= \frac{a\cos(\lambda-\delta)}{\sqrt{1-e^2\sin^2(\lambda-\delta)}} = m\cos(\lambda-\delta), \\ y_2 &= \frac{a(1-e^2)\sin(\lambda-\delta)}{\sqrt{1-e^2\sin^2(\lambda-\delta)}} = m(1-e^2)\sin(\lambda-\delta) \end{aligned}\right\} \ldots\ldots\ldots (2)$$

Moreover since LP is perpendicular to the tangent at P, it coincides with the normal, and hence LP is inclined at an angle $\lambda - \delta$ to the major axis; but the tangent of this angle is $\frac{y_1+y_2}{x_1+x_2}$;

$$\therefore \quad \tan(\lambda-\delta) = \frac{y_1+y_2}{x_1+x_2} = (1-e^2)\frac{\sin\lambda + \sin(\lambda-\delta)}{\cos\lambda + \cos(\lambda-\delta)}, \text{ by } (1, 2),$$

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$\therefore\ e^2\{\sin\lambda+\sin(\lambda-\delta)\}.\cos(\lambda-\delta)=\sin\lambda.\cos(\lambda-\delta)-\cos\lambda.\sin(\lambda-\delta)=\sin\delta.$

Hence writing $\sin\lambda$, $\cos\lambda$, and δ for $\sin(\lambda-\delta)$, $\cos(\lambda-\delta)$ and $\sin\delta$, we have with considerable accuracy

$$\delta = 2e^2\sin\lambda\cos\lambda = e^2\sin2\lambda \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

This is the correction for the latitude in circular measure; to reduce it to minutes multiply it by $\frac{180\times60}{\pi}$;

$$\therefore\quad \delta' = \frac{10800}{\pi}e^2\sin2\lambda = 22{\cdot}93\sin2\lambda \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The latitude of London being about 51° 30′ N., we find $\delta'=22'\ 20''$, hence the latitude of the place at the greatest distance from London is $\lambda-\delta=51°\ 7'\ 40''$ S.

Again, let D be the distance between London and its antipodes, and $D+\Delta$ that between London and P. We easily find

$$D = 2m\{\cos^2\lambda+(1-e^2)^2\sin^2\lambda\}^{\frac{1}{2}}.$$

Also, expanding $\cos(\lambda-\delta)$ and $\sin(\lambda-\delta)$ in powers of δ, retaining quantities of the second order and substituting for δ from (3), we have

$$a\{1-e^2\sin^2(\lambda-\delta)\}^{-\frac{1}{2}}=m(1-\tfrac{1}{2}e^4\sin^22\lambda),$$

$$\therefore\ x_2=m\{\cos\lambda+e^2\sin2\lambda\sin\lambda-e^4\sin^22\lambda.\cos\lambda\},$$

and $$y_2=m(1-e^2)\{\sin\lambda-e^2\sin2\lambda.\cos\lambda-e^4\sin^22\lambda.\sin\lambda\};$$

$$\therefore\ D+\Delta=\sqrt{(x_1+x_2)^2+(y_1+y_2)^2}=D+\tfrac{1}{8}De^4\sin^22\lambda=D+\tfrac{1}{4}ae^4\sin^22\lambda,$$

$$\therefore\quad \Delta=\tfrac{1}{4}ae^4\sin^22\lambda=\cdot044\sin^22\lambda \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where Δ, the excess of the distance between London and P, over that between London and its antipodes, is expressed as a decimal of a mile. Since $\lambda=51°\ 30'$, we have Δ equal to about 73 or 74 yards. This quantity being so extremely small, we may consider Captain Cook's remark as substantially correct. Indeed the assumption that the earth is accurately an ellipsoid of revolution, may, I should imagine, cause a greater error than 74 yards.

It is scarcely necessary to remark that the preceding results apply to any place, the latitude being λ.

Note. If, in finding the value of δ, we retain fourth powers of e, we shall have instead of (3) and (4)

$$\delta = e^2(1-e^2\cos^2\lambda)\sin2\lambda \ldots\ldots\ldots\ldots\ldots\ldots (A)$$

and $$\delta'=22{\cdot}93(1-\frac{2}{300}\cos^2\lambda)\sin2\lambda \ldots\ldots\ldots\ldots\ldots\ldots (B)$$

and when $\lambda=51°\ 30'$, there results $\delta'=22'\ 17''$, and $\lambda-\delta=51°\ 7'\ 43''$, which are only very slightly different from the above.

It is easily seen from (*B*) and (5) that the greatest deviation in latitude and distance will be about 22′ 51″ and 72 yards, corresponding to latitude 45° nearly.

Solutions were received from Messrs. Henry Blythe, Mr. Jeffery's Academy, Broom Hall, Shooter's Hill; R. Harley, Northwich; and Nicholas Smyth, Galway.

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LXV. *Mr. Fenwick.*

Prove that the locus of the centres of the innumerable surfaces of the second order that have eight points common in space, is a surface of the second order, and thence deduce the corresponding locus in plano.

The locus in space, it will be seen, is the intersection of *two* surfaces of the second order, that is, two hyperboloids of one sheet (see page 81.)—S.F.

[SOLUTION. *Mr. Thomas Weddle.*]

Let $X = 0 \ldots (1)$ $X' = 0 \ldots (2)$ be the equations of two of the surfaces passing through the eight points, then will

$$X + \lambda X' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

represent all the surfaces, λ being arbitrary; let also

$$U = 0, \quad V = 0, \quad W = 0 \ldots\ldots\ldots\ldots\ldots (4)$$

and

$$U' = 0, \quad V' = 0, \quad W' = 0 \ldots\ldots\ldots\ldots\ldots (5)$$

be the equations for determining the centres of (1) and (2) respectively. Then will

$$U + \lambda U' = 0, \quad V + \lambda V' = 0, \quad W + \lambda W' = 0 \ldots\ldots (6)$$

determine the position of the centre of (3), and the elimination of the variable λ will give the locus of the centre. The elimination being effected, we have

$$UV' - U'V = 0, \quad UW' - U'W = 0, \quad VW' - V'W = 0 \ldots (7)$$

and any one of the equations (7) is a consequence of the other two. We hence infer that in general the locus is a curve which is the intersection of two surfaces of the second order.

The preceding investigation will apply to curves of the second order passing through four points in a plane, if (1, 2) represent two of the curves, and those equations be rejected which contain W or W'. We shall find finally, $UV' - U'V = 0$, from which we see that the locus is a curve of the second order.

We may observe that the locus would be a surface of the third order, if the surfaces were restricted to pass through *seven* points only.

It may not perhaps be much out of place to notice the following theorem here, as it is somewhat connected with the above, as well as with various discussions in the "Mathematician."

Let diameters conjugate to a given plane be drawn to the innumerable surfaces of the second order passing through eight points in space, the locus of these diameters is a surface of the second order.

Let $U = 0$, $V = 0$, and $U' = 0$, $V' = 0$, be the equations to the diameters of (1) and (2) respectively, which are conjugate to the given plane; then will the diameter of (3) conjugate to the given plane be denoted by

$$U + \lambda U' = 0, \quad V + \lambda V' = 0.$$

The elimination of λ gives $UV' - U'V = 0$, for the locus which is therefore a surface of the second order.

Messrs. Henry Blythe, of Mr. Jeffery's Academy, Shooter's Hill, Woolwich; Nicholas Smyth, Galway; and R. H. Wright, London, sent also solutions. Mr. G. W. Hearn's investigation is given at page 116.

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LXVI. *Mr. G. W. Hearn, R.M. College, Sandhurst.*

Required the mutual action of two indefinitely small plane closed voltaic currents.

[SOLUTION. *Mr. G. W. Hearn, the Proposer.*]

Let the origin be first taken at a point in one of the closed currents, and let x, y, z be the coordinates of any point in the other closed current, δs being an element of that current, and $\delta s'$ an element of the other. Then according to *Ampère's* theory, if

$$A = \int \frac{ydz - zdy}{r^3},$$

$$B = \int \frac{zdx - xdz}{r^3},$$

$$C = \int \frac{xdy - ydx}{r^3},$$

(the integrals extending throughout the closed currents) we have for the forces on the element $\delta s'$

$$X = \tfrac{1}{2}\rho\rho'\delta s'\{C\cos\beta' - B\cos\gamma'\},$$
$$Y = \tfrac{1}{2}\rho\rho'\delta s'\{A\cos\gamma' - C\cos\alpha'\},$$
$$Z = \tfrac{1}{2}\rho\rho'\delta s'\{B\cos\alpha' - A\cos\beta'\},$$

in which ρ, ρ' are the intensities, and α', β', γ' the direction angles of $\delta s'$.

Now the values of A, B, C for an indefinitely small current are

$$C = -\frac{a}{r^3}\left\{\cos n - \frac{3pz}{r^2}\right\},$$

$$B = -\frac{a}{r^3}\left\{\cos m - \frac{3py}{r^2}\right\},$$

$$A = -\frac{a}{r^3}\left\{\cos l - \frac{3px}{r^2}\right\},$$

where a is the area; p the perpendicular on its plane from the origin; and x, y, z the coordinates of its centre, so that

$$r^2 = x^2 + y^2 + z^2,$$
$$p = x\cos l + y\cos m + z\cos n,$$

l, m, n being the direction angles of p.

Now suppose the origin to be changed to the point which is the centre of the closed current, whose action on $\delta s'$ we have just been calculating, then

$$\cos\alpha' = \frac{dx}{ds'}, \quad \cos\beta' = \frac{dy}{ds'}, \quad \cos\gamma' = \frac{dz}{ds'}.$$

Hence

$$X = \tfrac{1}{2}\rho\rho' a\,\delta s'\left\{\cos n\frac{1}{r^3}\cdot\frac{dy}{ds'} - \cos m\,\frac{1}{r^3}\cdot\frac{dz}{ds'} - \frac{3p(zdy - ydx)}{r^5 ds'}\right\};$$

and therefore the whole force in x is

$$\Sigma X = \tfrac{1}{2}\rho\rho' a\left\{\cos n\int\frac{dy}{r^3} - \cos m\int\frac{dz}{r^3} - 3\int\frac{p(zdy - ydz)}{r^5}\right\}.$$

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I shall not have space to enter into the detail of the process (which however is very simple) of finding the integrals, but I give the following as results,

$$\int \frac{dx}{r^n} = \frac{na}{R^{n+2}}\left(b\cos y - c\cos\beta\right),$$

$$\int \frac{dy}{r^n} = \frac{na}{R^{n+2}}\left(c\cos a - a\cos\gamma\right),$$

$$\int \frac{dz}{r^n} = \frac{na}{R^{n+2}}\left(a\cos\beta - b\cos a\right),$$

$$\int \frac{x dy - y dx}{r^n} = -\frac{a}{R^n}\left\{(n-2)\cos\gamma - \frac{ncp}{R^2}\right\}$$

$$\int \frac{y dz - z dy}{r^n} = -\frac{a}{R^n}\left\{(n-2)\cos a - \frac{nap}{R^2}\right\}$$

$$\int \frac{z dx - x dz}{r^n} = -\frac{a}{R^n}\left\{(n-2)\cos\beta - \frac{nbp}{R^2}\right\}$$

$$\int \frac{p(x dy - y dx)}{r^5} = -\frac{a'}{R^5}\left\{2p\cos n' + p'\cos n - \frac{5cpp'}{R^2}\right\}$$

$$\int \frac{p(y dz - z dy)}{r^5} = -\frac{a'}{R^5}\left\{2p\cos l' + p'\cos l - \frac{5app'}{R^2}\right\}$$

$$\int \frac{p(z dx - x dz)}{r^5} = -\frac{a'}{R^5}\left\{2p\cos m' + p'\cos m - \frac{5bpp'}{R^2}\right\}$$

in which dashed letters signify the same quantities with respect to the other current.

By means of these we have for the whole force in the direction of x

$$\Sigma X = \frac{3}{2}\cdot\frac{\rho\rho' aa'}{R^5}\left\{\frac{5app'}{R^2} - a\cos\phi - p\cos l' - p'\cos l\right\};$$

where $\cos\phi = \cos l\cos l' + \cos m\cos m' + \cos n\cos n'$, and similar quantities for ΣY and ΣZ.

ϕ is the angle of inclination of the planes of the currents, and if A, B be the angles of their inclination to R, the line joining their centres, we have by taking the square root of the sum of the squares of the components, and putting $p = R\sin A$, $p' = R\sin B$, the following value of the resultant,

$$\frac{3}{2}\cdot\frac{\rho\rho' aa'}{R^4}\left\{(\cos\phi - 2\sin A\sin B)^2 + \sin^2 A + \sin^2 B + \sin^2 A\sin^2 B\right\}^{\frac{1}{2}}.$$

When the planes of the currents are each at right angles to the line joining their centres, $A = B = 90°$ and $\phi = 0$, and the preceding becomes $\frac{3\rho\rho' aa'}{R^4}$.

When the planes of the currents coincide, then $A = 0$, $B = 0$, $\phi = 0$, and the action is $\frac{3\rho\rho' aa'}{2R^4}$.

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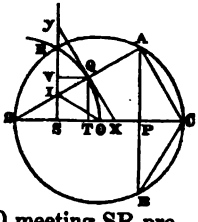
The action is therefore double as much in the former as in the latter case.

Several curious properties readily follow from the preceding, of which I will instance one. The plane in which the resultant lies divides the angle between the planes through pR, $p'R$ into two angles whose sines are to each other as the tangents of the angles which the planes of the closed currents make with their mutual distance R.

LXVII. *Mr. Matthew Collins, Limerick.*

Determine geometrically an arc of a given circle such that the excess of its chord above the chord of half the arc may be a maximum.

[SOLUTION. *By the Proposer.*]



Analysis. Suppose that the excess of the chord AB above the chord of half the arc AC is a maximum. Draw the diameter COD intersecting AB at right angles in P. Join AD, and from D with the radius of the circle DO describe the arc OQR intersecting the line AD in Q, and the circle in R. From R draw the line RS perpendicular to CD bisecting DO in S and intersecting DA in I: draw QT and QV perpendicular to CD and SR: through Q draw the tangent XQY at right angles to DQ meeting SR produced in Y and DO in X, and draw IO. Then the triangles DIO and ABC being equiangular,

$$\text{DO or DQ} : \text{DI} :: \text{AB} : \text{AC},$$

$$\text{or, by division,} \quad \text{QI} : \text{DI} :: \text{AB—AC} : \text{AC};$$

$$\text{but} \quad \text{QI} : \text{DI} :: \text{ST} : \text{SD};$$

$$\therefore \quad \text{SD} : \text{ST} :: \text{AC} : \text{AB—AC}.$$

Now SD is given, being equal to half the radius OD; hence AB—AC will be a maximum when ST.AC is a maximum; but by similar triangles DQ : QT :: DC : AC, and DC = 2DQ; therefore AC = 2QT, and hence ST.TQ or VQ.QT must be a maximum, which is known to be the case when ST = TX (Simpson's Max. and Min., Theorem VIII), or when the parts QX and QY of the tangent are equal. Lastly since

$$\text{DQ}^2 = \text{DT.DX} = (\text{DS}+\text{ST})(\text{DS}+2\text{ST}) = \text{DS}^2+3\text{DS.ST}+2\text{ST}^2;$$

$$\therefore \quad \left(\frac{3}{2}\text{DS} + \text{ST}\right)\text{ST} = \frac{\text{DQ}^2 - \text{DS}^2}{2};$$

and since DQ and DS are both given magnitudes, the line ST is easily determined, and thence the arc ACB.

Messrs. Hugh Godfray, Jersey; and W. H. Levy, Shalbourne, sent solutions to this exercise.

LXVIII. *Mr. John Riddle, Nautical School, Greenwich Hospital.*

An observer stands in such a position as to see three angles of a square tower standing on a horizontal plane, and he observes the angles of elevation of the three upper corners to be, for the nearest $\alpha°$, and for the other two $\beta°$ and $\gamma°$; and his nearest distance from the tower is m feet. Required the length of the side of the square, and the height of the tower.

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[FIRST SOLUTION. *Mr. Thomas Weddle, Newcastle; and similarly by Mr. Hugh Godfray, Jersey.*]

Let P be the position of the observer, A, B, C the three lower corners of the tower, (A being nearest the observer), and L, M, N the corresponding upper corners. Put PA $= m$, PB $= n$, PC $= p$, $x =$ AB or AC the side of the square, $h =$ height of the tower, and $\theta =$ ∠ BAP. Then in the right angled triangles PAL, PBM, and PCN we have

$$h = m \tan \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$n = h \cot \beta, \quad p = h \cot \gamma \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and these equations determine h, n and p.

In the triangle BAP we have, $2mx \cos \theta = m^2 + x^2 - n^2 \ldots\ldots\ldots (3)$

and, in the triangle CAP, $2mx \cos \text{CAP} = m^2 + x^2 - p^2 \ldots\ldots (4)$

But the angle BAC being a right angle, we have CAP $= \frac{3}{2}\pi - \theta$, hence (4) becomes

$$-2mx \sin \theta = m^2 + x^2 - p^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Add the squares of (3, 5),

$$\therefore \; 4m^2x^2 = (m^2 + x^2 - n^2)^2 + (m^2 + x^2 - p^2)^2 \ldots\ldots\ldots (6)$$

Expand this equation, and solve the resulting quadratic in x^2,

$$\therefore \; x^2 = \tfrac{1}{2}(n^2 + p^2) - \tfrac{1}{2}\sqrt{4m^2n^2 + 4m^2p^2 + 2n^2p^2 - 4m^4 - n^4 - p^4}$$

$$= \tfrac{1}{2}(n^2 + p^2) - 2\Delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

where Δ is the area of a triangle whose sides are $m\sqrt{2}$, n and p. The reason why *minus* is prefixed to the radical is that x being less than either n or p, x^2 is evidently less than $\frac{1}{2}(n^2 + p^2)$.

The preceding result might have been obtained by aid of question 1743, proposed by Mr. Rutherford in the Lady's and Gentleman's Diary for 1845, and answered in the last Diary, p. 59.

[SECOND SOLUTION. *Mr. James Reid, Prior's Salford; and similarly by Messrs. S. Bills, Hawton; and J. Liddell, Gentleman Cadet, R.M. Academy.*]

Let ABCD be the base of the tower, and O the place of the observer, O being obviously in the angle contained by BA, CA, produced.

Through O in the plane AD draw Ox, Oy parallel respectively to BA, AC, and let CA meet Ox in M.

Put OA $= m$, OB $= m_1$, OC $= m_2$, and let $h =$ height of the tower; then $h = m \tan \alpha$.

Hence $m_1 = h \cot \beta = m \tan \alpha \cot \beta$,

and $m_2 = h \cot \gamma = m \tan \alpha \cot \gamma$,

are given lines.

Again, let OM $= x$, MA $= y$, AD $= z$; then

$$m^2 = x^2 + y^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$m_1^2 = (x+z)^2 + y^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$m_2^2 = x^2 + (y+z)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

By (1, 2), $m_1^2 - m^2 = 2xz + z^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$

By (1, 3), $m_2^2 - m^2 = 2yz + z^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$

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Substituting the values of x and y derived from (4, 5) in (1), and resolving for z, we have

$$z = \pm \tfrac{1}{2}\sqrt{2[m_1^2+m_2^2 \pm \sqrt{\{(m_1^2+m_2^2)^2-2(m_1^2-m^2)^2-2(m_2^2-m^2)\}}]^{\frac{1}{2}}},$$

which is the length of the side of the square.

The double sign arises from the generality of the investigation, as if the question had been—*Given a point and the distances from it to three of the angles of a square, to determine the magnitude and position of the square.*

The ambiguity is got rid of by referring to the conditions of the problem. The value of z cannot be negative, for then O would not lie in the exterior angle formed by producing CA, BA; wherefore the *positive* sign without the vinculum must be taken.

Again, join BC; then the angle BOC is acute, and hence BC^2 is less than $(m_1^2+m_2^2)$, or z is less than $\frac{1}{2}\sqrt{\{2(m_1^2+m_2^2)\}}$. It follows, therefore, that the *negative* sign *within* the vinculum must be used.

Good solutions were received from Dr. Burns, Rochester, and the Proposer.

LXIX. *Mr. J. W. Elliott, Greatham.*

If from a point P in a conic section, a tangent PO be drawn to meet the axis major AB in O, and the chords AP, BP to meet a perpendicular through O in Q, Q': then QO = OQ'.

[First Solution. *Mr. H. Godfray, David Place, St. Mark's Road, Jersey; and similarly by Mr. Thomas Weddle, Newcastle.*]

This theorem admits of greater generality than as proposed in the exercise; it may be thus enunciated:—If from a point P in a conic section, a tangent PO be drawn to meet any diameter AB in O, and the chords AP, BP to meet a conjugate to AB in Q, Q'; then QO = Q'O.

For draw the ordinate PM, then (*Davies's Hutton*, II, page 138, *cor.* 1, *for the ellipse, and* page 164, *cor.* 1, *for the hyperbola*) AB is divided harmonically in M and O:

$$\therefore \quad AM : AO :: BM : BO.$$

But by similar triangles,

$$AM : AO :: PM : QO, \text{ and } BM : BO :: PM : Q'O;$$
$$\therefore \quad QO = Q'O.$$

The same property holds for the parabola by considering the chord through A to be another diameter through P.

The solution of *Mr. Samuel Bills, Hawton*, was analogous to the preceding, AB being the *major axis*, as in the exercise.

[Second Solution. *Mr. G. W. Hearn; and similarly by Dunelmensis, and Mr. James Reid, Prior's Salford.*]

Taking the origin at C the centre, we have $CO = \frac{a^2}{x'}$. Also

$$\frac{y}{y'} = \frac{x+a}{x'+a} \text{ equation to chord AP,}$$

$$\frac{y}{y'} = \frac{x-a}{x'-a} \quad \ldots\ldots\ldots\ldots \text{PB,}$$

x' and y' being coordinates of P.

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Put $\frac{a^2}{x'}$ for x in these equations, and we determine

$$y = \frac{ay'}{x'} \text{ from the first,}$$

$$y = \frac{ay'}{x'} \ldots\ldots \text{ second.}$$

Therefore $QO = OQ' = \frac{ay'}{x'}$.

[THIRD SOLUTION. *By the Proposer.*]

Draw the chord PP′ meeting AB perpendicularly in O′, and join AP, AP′, BP, BP′, and OP′. Then since OP, OP′ are tangents to the conic section, and APBP′ an inscribed quadrilateral, the intersections Q, Q′ (suppose) of the opposite sides of APBP′ range in a straight line with O, the pole of the polar PP′. Moreover $PO' = O'P'$, therefore QQ′ is parallel to PP′—(*Mathematician*, Vol. II., p. 39); and $AQ = AQ'$, $\therefore QO = OQ'$.

Mr. Elliott sent two other good solutions.

Correct and elegant solutions were also received from Dr. Burns, Rochester; Messrs. R. Harley, Northwich, Cheshire; J. Liddell, Gentleman Cadet, R. M. Academy; Lieut. St. Andrew St. John, R. E.; and R. H. Wright, London.

LXX. *Mr. Thomas Cotterill, Panton Square, Haymarket.*

If K be the area of a triangle ABC whose sides are a, b, c; R the radius of the circumscribed circle, and α, β, γ, points in the sides a, b, c, or in the sides produced; then K′ being the area of the triangle $\alpha\beta\gamma$, we shall have

$$\frac{A\beta.\,C\alpha.\,B\gamma + C\beta.\,A\gamma.\,B\alpha}{K'} = \frac{abc}{K} = 4R.$$

[FIRST SOLUTION. *By Dunelmensis; and similarly by Mr. R. Harley, Northwich, Cheshire.*]

The area of the triangle $A\beta\gamma = \frac{1}{2} A\beta.A\gamma.\sin A = \frac{A\beta.A\gamma}{bc}.K$; similarly,

$$\Delta B\alpha\gamma = \frac{Ba.B\gamma}{ac}.K, \quad \Delta C\alpha\beta = \frac{Ca.C\beta}{ab}.K.$$

When the points α, β, γ are in the sides of the triangle, the sum of these together with the $\Delta\alpha\beta\gamma$ or K′ is equal to K; therefore

$$K' = K - \frac{A\beta.A\gamma}{bc}.K - \frac{Ba.B\gamma}{ac}.K - \frac{Ca.C\beta}{ab}.K,$$

$$\text{or} \quad abc.\frac{K'}{K} = abc - a.\,A\beta.A\gamma - b.Ba.B\gamma - c.Ca.C\beta.$$

But $a = Ba + Ca$, $b = A\beta + C\beta$, $c = A\gamma + B\gamma$, and making these substitutions on the right side, we have it $= A\gamma.Ba.C\beta + B\gamma.Ca.A\beta$;

$$\therefore \quad \frac{A\gamma.Ba.C\beta + B\gamma.Ca.A\beta}{K'} = \frac{abc}{K} = 4R.$$

[SECOND SOLUTION. *Mr. Thomas Weddle.*]

Denote the triangles $A\beta\gamma$, $B\gamma a$, and $C a\beta$ by K_1, K_2, and K_3. The angles BAC and $\gamma A\beta$ being coincident, we have

$$\frac{K_1}{K} = \frac{A\beta.A\gamma}{bc} = \frac{A\beta.A\gamma(Ba+Ca)}{abc}; \text{ similarly } \frac{K_2}{K} = \frac{Ba.B\gamma(A\beta+C\beta)}{abc},$$

$$\text{and } \frac{K_3}{K} = \frac{Ca.C\beta(A\gamma+B\gamma)}{abc}. \text{ Also } \frac{K}{K} = \frac{(Ba+Ca)(A\beta+C\beta)(A\gamma+B\gamma).}{abc}$$

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Hence we at once obtain,

$$\frac{K - K_1 - K_2 - K_3}{K}, \text{ that is } \frac{K'}{K} = \frac{A\beta.Ca.B\gamma + C\beta.A\gamma.Ba}{abc};$$

$$\therefore \frac{A\beta.Ca.B\gamma + C\beta.A\gamma.Ba}{K'} = \frac{abc}{K} = \text{(Lady's Diary, for 1835, p. 55) } 4R.$$

[THIRD SOLUTION. *Mr. W. H. Levy, Shalbourne.*]

Let K_1, K_2 and K_3 be the areas of the triangles $A\beta\gamma$, $Ba\gamma$ and $Ca\beta$ respectively; then $2K = ab \sin C$; but $2R \sin C = c$; therefore $4RK = abc$. Again $A\beta.A\gamma \sin A = 2K_1$, or as before $A\beta.A\gamma.a = 4RK_1$. Similarly we have

$$Ba.B\gamma.b = 4RK_2, \quad Ca.C\beta.c = 4RK_3.$$

Now $K' = K - K_1 - K_2 - K_3$; therefore

$$4RK' = abc - A\beta.A\gamma.a - Ba.B\gamma.b - Ca.C\beta.c,$$

and substituting $Ba + Ca$ for a, $A\beta + C\beta$ for b, and $A\gamma + B\gamma$ for c, we find

$$4RK' = A\beta.Ca.B\gamma + C\beta.A\gamma.Ba;$$

$$\therefore \frac{A\beta.Ca.B\gamma + C\beta.A\gamma.Ba}{K'} = 4R.$$

If one of the points a is taken in the side CB produced, then we get,

$$A\beta.Ca.B\gamma - C\beta.A\gamma.Ba = 4RK',$$

and if the points β, γ, a range in a straight line, we have $K' = 0$, and consequently

$$A\beta.Ca.B\gamma = C\beta.A\gamma.Ba,$$

which is the well known relation between the segments of the sides of the triangle ABC made by a transversal $\beta\gamma a$.

The necessary modifications of the theorem for any position of the points a, β, γ present no difficulty.

Solutions were received from Messrs. Samuel Bills, Hawton; J. W. Elliott, Greatham; G. W. Hearn, Sandhurst; and R. H. Wright, London.

LXXI. *Mr. Thomas Cotterill, Panton Square, Haymarket.*

If the triangle K' taken as in last exercise be co-polar with the triangle K, S being the pole, and K_a, K_b, K_c be the areas of the triangles BCS, CAS, ABS respectively; then

$$\frac{2}{K'} + \frac{1}{K} = \frac{1}{K_a} + \frac{1}{K_b} + \frac{1}{K_c}.$$

[FIRST SOLUTION. *Mr. Thomas Weddle.*]

The triangles K_a and K_b have the side CS common, and the perpendiculars from B and A on this side are evidently proportional to $B\gamma$ and $A\gamma$; therefore

$$\frac{K_b}{K_a} = \frac{A\gamma}{B\gamma}. \quad \text{Similarly, } \frac{K_c}{K_a} = \frac{A\beta}{C\beta}, \text{ and } \frac{K_c}{K_b} = \frac{Ba}{Ca}. \quad \text{Moreover,}$$

$$K = K_a + K_b + K_c; \quad \text{hence}$$

$$\frac{K}{K_a} + \frac{K}{K_b} + \frac{K}{K_c} = 3 + \frac{K_b}{K_a} + \frac{K_c}{K_a} + \frac{K_a}{K_b} + \frac{K_c}{K_b} + \frac{K_a}{K_c} + \frac{K_b}{K_c}$$

$$= 3 + \frac{A\gamma}{B\gamma} + \frac{A\beta}{C\beta} + \frac{B\gamma}{A\gamma} + \frac{Ba}{Ca} + \frac{C\beta}{A\beta} + \frac{Ca}{Ba} \quad \ldots (1)$$

Moreover (Hutton's Course, Vol. II. p. 217, 12th edit.) $A\beta.Ca.B\gamma = B\beta.A\gamma.Ba = P$ (suppose), hence by the last exercise,

$$\frac{2K}{K'} = \frac{abc}{P} = \frac{(Ba + Ca)(A\beta + C\beta)(A\gamma + B\gamma)}{P} =$$

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$$\frac{A\beta.C\alpha.B\gamma + A\beta.C\alpha.A\gamma + A\beta.B\gamma.B\alpha + C\alpha.B\gamma.C\beta}{A\beta.C\alpha.B\gamma} +$$
$$\frac{C\beta.A\gamma.B\alpha + C\beta.A\gamma.C\alpha + C\beta.B\alpha.B\gamma + A\gamma.B\alpha.A\beta}{C\beta.A\gamma.B\alpha}$$
$$= 2 + \frac{A\gamma}{B\gamma} + \frac{B\alpha}{C\alpha} + \frac{C\beta}{A\beta} + \frac{C\alpha}{B\alpha} + \frac{B\gamma}{A\gamma} + \frac{A\beta}{C\beta} \quad \ldots\ldots (2)$$

Comparing (1, 2), we have

$$\frac{2K}{K'} + 1 = \frac{K}{K_a} + \frac{K}{K_b} + \frac{K}{K_c}.$$

Divide by K, $\quad \therefore \frac{2}{K'} + \frac{1}{K} = \frac{1}{K_a} + \frac{1}{K_b} + \frac{1}{K_c}.$

[SECOND SOLUTION. *Mr. G. W. Hearn.*]

Let $k_a = S\beta\gamma$, $k_b = S\alpha\gamma$, $k_c = S\alpha\beta$; then we have obviously

$$K_a + K_b + K_c = K \quad \ldots\ldots (1)$$
$$k_a + k_b + k_c = K' \quad \ldots\ldots (2)$$

But $k_a = \frac{1}{2}S\beta.S\gamma.\sin\beta S\gamma$, $K_a = \frac{1}{2}SB.SC \sin BSC$,
and the opposite vertical angles $\beta S\gamma$ and BSC are equal; therefore

$$k_a = \frac{S\beta}{SB}.\frac{S\gamma}{SC}.K_a;$$

but $\quad \frac{S\beta}{B\beta} = \frac{K_b}{K}, \quad \therefore \frac{S\beta + SB}{S\beta} = \frac{K}{K_b},$

or $\quad \frac{SB}{S\beta} = \frac{K}{K_b} - 1, \quad \therefore \frac{S\beta}{SB} = \frac{K_b}{K - K_b},$

and so of the others; therefore

$$k_a = \frac{K_a K_b K_c}{(K - K_b)(K - K_c)}.$$

Hence

$$K' = \frac{K_a K_b K_c}{(K-K_a)(K-K_b)(K-K_c)}\{K-K_a + K-K_b + K-K_c\}$$
$$= \frac{2K K_a K_b K_c}{(K-K_a)(K-K_b)(K-K_c)}, \text{ by (1) and (2)}$$
$$\therefore \frac{2}{K'} = \frac{K^3-(K_a+K_b+K_c)K^2+(K_aK_b+K_bK_c+K_cK_a)K-K_aK_bK_c}{K.K_aK_bK_c}$$
$$= \frac{1}{K_a} + \frac{1}{K_b} + \frac{1}{K_c} - \frac{1}{K}, \text{ since } K_a + K_b + K_c = K;$$
$$\therefore \frac{2}{K'} + \frac{1}{K} = \frac{1}{K_a} + \frac{1}{K_b} + \frac{1}{K_c}.$$

The analogous stereometrical problem is also easily derived in a similar manner.

We find $\quad k_a = \frac{K_aK_bK_cK_d}{(K-K_b)(K-K_c)(K-K_d)},$

and $\quad K_1 = \frac{3KK_aK_bK_cK_d}{(K-K_a)(K-K_b)(K-K_c)(K-K_d)},$

which is equivalent to

$$\frac{3}{K'} - \frac{1}{K} = K\Sigma\frac{1}{K_aK_b} - \Sigma\frac{1}{K_a} \quad \ldots\ldots (a)$$

and the theorem may be thus enunciated.

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Let a point be taken within a triangular pyramid, and let lines be drawn from the angular point through the assumed point to intersect the opposite faces a, b, c, d; and let K′ be the volume of the pyramid whose vertices are the points of intersection, K the volume of the whole pyramid divided into the four pyramids K_a, K_b, K_c, K_d, by the lines through the assumed point; then will

$$\frac{3}{K'} - \frac{1}{K} = K\Sigma \frac{1}{K_a K_b} - \Sigma \frac{1}{K_a}.$$

[THIRD SOLUTION. *Mr. J. W. Elliott, Greatham.*]

Denote the triangles $C\alpha\beta$, $A\beta\gamma$, $B\gamma\alpha$ by λ_1, λ_2, λ_3, and $S\alpha\beta$, $S\beta\gamma$, $S\gamma\alpha$ by l_1, l_2, l_3, respectively: then obviously

$$\frac{K}{K_c} = \frac{\lambda_1}{l_1}, \quad \frac{K}{K_a} = \frac{\lambda_2}{l_2}, \quad \frac{K}{K_b} = \frac{\lambda_3}{l_3}; \text{ or}$$

$$\frac{1}{K_a} + \frac{1}{K_b} + \frac{1}{K_c} = \left(\frac{\lambda_1}{l_1} + \frac{\lambda_2}{l_2} + \frac{\lambda_3}{l_3}\right)\frac{1}{K} \ldots\ldots\ldots (1)$$

Now by a process similar to that in the last exercise, it can be shown, that

$$\frac{\lambda_1}{l_1} + \frac{\lambda_2}{l_2} + \frac{\lambda_3}{l_3} = 1 + \frac{abc}{A\beta.C\alpha.B\gamma},$$

and the triangle K′ being co-polar with the triangle K, we have by the last exercise

$$\frac{2A\beta.C\alpha.B\gamma}{abc} = \frac{K'}{K}, \text{ or } \frac{2}{K'} + \frac{1}{K} = \frac{1}{K}\left(1 + \frac{abc}{A\beta.C\alpha.B\gamma}\right).$$

Hence (1)

$$\frac{1}{K^a} + \frac{1}{K_b} + \frac{1}{K_c} = \frac{2}{K'} + \frac{1}{K}.$$

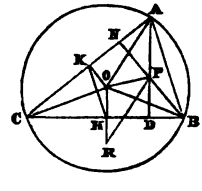
Mr. Bills favored us with a good solution.

LXXII. *Mr. Rutherford.*

If three equal forces be represented in magnitude and direction by the three lines drawn from the angular points of a plane triangle to the centre of the circumscribed circle, their resultant will be represented in magnitude and direction by the straight line which joins the centre of the circumscribed circle and the point of intersection of the perpendiculars from the angles on the opposite sides.

[FIRST SOLUTION. *William Burns, L.L.D., Rochester.*]

Let ABC be the triangle, O the centre of the circumscribed circle, P the point of intersection of the perpendiculars AD, BN; and draw OA, OB, OC, OP. Draw also OH, OK perpendicular to BC, CA, and join HK. Since AK = KC, and CH = HB, therefore HK is parallel to AB, and AB equal to 2HK. Also the triangles APB and HOK being equiangular, BP = 2OK. Hence if the force OC be resolved into forces represented in magnitude and direction by OK, KC, and OA into two represented by OK, KA; then KC, KA being equal and opposite, 2OK = BP is the resultant of OA, OC. Also if OB be resolved into the two OP, PB, then since BP and PB are equal and opposite, OP is evidently the direction and magnitude of the resultant of the three forces OA, OB, OC.



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[SECOND SOLUTION. *Mr. Thomas Weddle; and similarly by Messrs. S. Bills, and W. H. Levy, Shalbourne.*]

Draw OH and AD perpendicular to BC: on AD take AP = 2OH, and draw OP; then P is the point of intersection of the perpendiculars from A, B, C on the opposite sides (Mathematical Companion, No. V, page 48). Now the force OA is equivalent to the forces PO and PA; OC to OH and HC; and OB to OH and HB; hence the three forces OA, OB, OC are equivalent to the forces PO, PA, 2OH, HC and HB acting at O. Of these PA and 2OH are equal and opposite, and so are HC and HB; hence the forces OA, OB, OC are equivalent to the single force OP, that is, the force OP is the resultant of the three equal forces OA, OB, OC.

[THIRD SOLUTION. *Mr. J. W. Elliott, Greatham, Stockton-on-Tees; and similarly by Messrs. J. Reid, Prior's Salford; H. Godfray, Jersey; R. Harley, Northwich; and H.R.L.*]

Let ABC be a plane triangle, O the centre of its circumscribed circle, and AO, BO, CO the lines representing the three equal forces in magnitude and direction. Draw OHR perpendicular to BC, and make OR = 2OH; then OR represents in magnitude and direction the resultant of BO, CO, as is well known.

Again, complete the parallelogram AORP, and draw the diagonal OP, which will represent in magnitude and direction the resultant of the forces OA, OR, that is, of the three forces AO, BO, CO.

It now only remains to show that P is the point of intersection of perpendiculars from A, B, C upon the opposite sides.

Since CO = OB, OR bisects the angle COB and intersects BC at right-angles in H; and therefore AP being parallel to OR by construction, coincides with a perpendicular from A upon BC. Moreover AP=OR =2OH; hence P is the point of intersection of perpendiculars from A,B,C. Vide Lady's and Gentleman's Diary, for 1843, p. 60.

[FOURTH SOLUTION. *By Dunelmensis.*]

Let A be the vertex of the plane triangle; BC the base; O the centre of the circumscribed circle, and R its radius. Then resolving the forces parallel and perpendicular to the base BC, we have for the component forces of

$$\begin{aligned}
&(OA)\ldots. R\sin(B-C) \text{ and } R\cos(B-C),\\
&(OB)\ldots. R\cos(90°-A) \text{ and } -R\sin(90°-A),\\
&(OC)\ldots -R\cos(90°-A) \text{ and } -R\sin(90°-A);\\
&\therefore\ \Sigma(X) = R\sin(B-C)\\
&\quad\ \Sigma(Y) = R\cos(B-C) - 2R\cos A
\end{aligned}\Bigg\}\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Again, since AP = 2OH, the component forces of OP are evidently,

$$R\sin(B-C), \text{ and } R\cos(B-C) - 2R\cos A,$$

which are identical with $\Sigma(X)$ and $\Sigma(Y)$ by (1); hence the resultant is represented, both in magnitude and direction, by the line OP.

Messrs. Hearn and Wright also sent good solutions of this exercise.

LXXIII. *Mr. Fenwick.*

Let the diagonals AC, BD of a parallelogram ABCD intersect in O, and through O draw EF, GH parallel respectively to the sides AB, BC. In EF take any point E, and let lines EAH, EDG meet GH in H and G. Then drawing the lines HB, GC, they will tend to a point F in EF, and EFGH will be a parallelogram. The property also holds for other lines through the point O, but not for *any* two lines; specify the failing cases.

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[SOLUTION. *Mr. Thomas Weddle.*]

Take OF and OH for axes, then, if $2a = AB$, $2b = BC$, and $OE = c$, the points A, B, C, D and E will be denoted by $(-a, b)$, (a, b), $(a, -b)$, $(-a, -b)$, and $(-c, 0)$. Hence,

the equation of EH is $y = \frac{b}{c-a}(x+c)$ ..... (1)

.... EG.. $y = \frac{-b}{c-a}(x+c)$ ..... (2)

Put $x = 0$, in (1, 2), and the resulting values of y are OH and $-OG$ respectively

$$\therefore \quad OH = OG = \frac{bc}{c-a} \quad \ldots\ldots (3)$$

Hence the equation of HB is

$$y - \frac{bc}{c-a} = \left(b - \frac{bc}{c-a}\right) x \div a,$$

which reduces to $y = \frac{-b}{c-a}(x-c)$ ..... (4)

Similarly the equation of GC is

$$y + \frac{bc}{c-a} = \left(\frac{bc}{c-a} - b\right) x \div a, \text{ or } y = \frac{b}{c-a}(x-c) \ldots\ldots (5)$$

Now (4) and (5) both pass through the point $(c, 0)$ in EF, hence HB and GC converge to a point F in EF. Also EH and GF are parallel (1, 5), and so are EG and HF (2, 4); consequently, EGHF is a parallelogram.

Suppose now that EF and GH are any two lines through O, and let it be required to find the conditions that the properties may still be true. Take OC and OB for axes, and denote the points A, B, C, D, E and H by $(-a, 0)$, $(0, \beta)$, $(a, 0)$, $(0, -\beta)$, (x_1, y_1), and (x_2, y_2).

The equation of EA is $y = \frac{y_1}{x_1 + a}(x + a)$ ..... (6)

and that of ED, $y + \beta = \frac{y_1 + \beta}{x_1} x$ ..... (7)

Since (6) passes through H or $(x_2 y_2)$, we have

$$\frac{y_2}{y_1} = \frac{x_2 + a}{x_1 + a} \quad \ldots\ldots (8)$$

Also the equation of HB, passing through H and B, is

$$y - \beta = \frac{y_2 - \beta}{x_2} x \quad \ldots\ldots (9)$$

and since HB and ED are parallel, we have (7, 9),

$$\frac{y_2 - \beta}{y_1 + \beta} = \frac{x_2}{x_1} \quad \ldots\ldots (10)$$

Now from (7, 10) it appears that ED passes through the point $(-x_2, -y_2)$, but this point is evidently in HOG, and hence it must be the point G. The equation of GC is therefore

$$y = \frac{y_2}{x_2 + a}(x - a) \quad \ldots\ldots (11)$$

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Now, by (6, 11, 8) EA and GC are parallel, and HB and ED being so too, EGFH is a parallelogram. Also it appears from (9, 10) and (11, 8), that HB and GC both pass through the point $(-x_1,-y_1)$ which is evidently on EOF.

Hence, if the conditions (8, 10) be satisfied, the properties enunciated will be true, so that the point E or $(x_1\ y_1)$ being given, the point H or $(x_2 y_2)$ may be found, and thence the straight line GH will be determined. It may be observed, that supposing EF to be given in position, the straight line GH will vary with the different positions of the point E. To show this, we observe that since EF is given in position, we have $\frac{y_1}{x_1} = \text{constant} = n$ (suppose). Eliminate x_1, y_1 from this and (8, 10), and the resulting equation, which is of the second order, is the locus of H. It is

$$ay_2(y_2 - nx_2 - \beta) + \beta x_2(y_2 - nx_2 - na) = 0 \quad \ldots\ldots\ldots (12)$$

Hence the locus of H, not being in general a straight line, the straight line GH will vary with the variation of the position of E on EF. I say *in general*, for if $\beta = na$, or $n = \frac{\beta}{a}$, then (since the straight line $ay_2 - \beta x_2 - a\beta = 0$ does not pass through O), (12) becomes $ay_2 + \beta x_2 = 0$, which is the equation of the straight line through O, parallel to BC, also $\frac{y_1}{x_1} = \frac{\beta}{a}$, or $ay_1 - \beta x_1 = 0$, and EF is parallel to AB. Hence the lines first considered are the only ones in which the points E and H may vary without the lines themselves varying.

It is evident that this last discussion includes the former, which might consequently have been omitted from this proof.

MATHEMATICAL EXERCISES—*(continued.)*

80. *By Juvenis.*

A merchant bought a cask of spirits for £48, and sold a quantity exceeding three-fourths of the whole by two gallons at a profit of £25 per cent. He afterwards sold the remainder at such a price as to clear £60 per cent by the whole transaction; and had he sold the whole quantity at the latter price he would have gained £175 per cent: find the number of gallons contained in the cask.

Note. This exercise is to be solved by the principles of arithmetic.

81. *Mr. R. Harley, Northwich.*

From each angle of a quadrilateral ABCD inscribed in a circle let a line be inflected to the opposite diagonal, so that it and the other diagonal may make equal angles with the adjacent sides: then will the areas of the triangles ABC, ADC, ABD, BCD, into which the quadrilateral is divided by its diagonals, be to each other respectively as the four inflected lines.

82. *By Geometricus.*

Given the sum of the two sides of a plane triangle, the radius of the inscribed circle, and either of the angles at the base, to construct the triangle.

83. *Mr. James Reid, Prior's Salford.*

If from the point in which a normal to an ellipse meets the axis major a straight line be drawn at right angles to the normal, the part of either of the focal distances intercepted between this line and the curve will be an harmonic mean between the focal distances.

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84. *Mr. John Riddle, Nautical School, Greenwich Hospital.*

A statue a feet high placed upon a column subtended an angle of $\theta°$ at a certain distance from the base, and b feet farther off it subtended an angle of $\phi°$; find the height of the column and distance of the observer from it at first.

85. *By Penumbra.*

There are given three lines SP, SQ, SR and two points A, B, all in the same plane. Let any triangle PQR be described having its angular points upon the three given lines, and two of its sides passing through the given points; then, whatever be this triangle, its third side will pass through a given point.

86. *By Amicus.*

In every tetrahedron, the sum of the squares of the six edges is equal to four times the sum of the squares of the lines which join the middles of the opposite edges.

[Our readers are requested to furnish any other properties of the figure, either new or little known, in order to the formation of a general collection of such properties.]

87. *Mr. Thomas Weddle.*

An ellipse can be divided geometrically into n equal sectors when n is such, that a regular polygon of n sides can be described geometrically: required proof.

88. *Mr. Fenwick.*

The equation in general of a surface of the second order when referred to any three coordinate planes, involves nine independent coefficients; if, however, the planes of reference be tangential to the surface, three of the nine coefficients will be functions of the remaining six. It is required to express the equation of the surface in terms of the six coefficients.

89. *Mr. Thomas Weddle.*

Sum the series

$$\operatorname{cosec}^{-1}\sqrt{[(1^2+1)(2^2+1)]}+\operatorname{cosec}^{-1}\sqrt{[2^2+1)(3^2+1)]}\ldots\ldots$$
$$+\operatorname{cosec}^{-1}\sqrt{[(n^2+1)\{(n+1)^2+1\}]};$$

also find a series, the sum of which ad infinitum, shall be a square, the sum of n terms a square, and the sum (ad inf.) of the series remaining after rejecting the n terms, also a square.

90. *Mr. Fenwick.*

Investigate the locus of the centres of surfaces of the second order inscribed in a given parallelopiped.

91. *Mr. G. W. Hearn, R. M. College.*

A wire ARM bent into the form of a spiral moves in a vertical plane about the pole C, the machine being so counterpoised that C is also the centre of gravity of the revolving mass; CR is a fixed vertical straight rod on which a small ring R slides, the spiral wire also passing through the ring. Find the form of the spiral and other circumstances, that the vertical motion of the ring may be uniform.

92. *By the same Gentleman.*

A rough horizontal plane is constrained to revolve round a vertical axis with a constant angular velocity, and a uniform cylinder is placed with its base upon it; if the friction be sufficient to prevent sliding, find the ratio of the diameter of the cylinder to its height, so that it may just upset.

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THE MATHEMATICIAN.

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ON THE CIRCULAR SECTIONS OF SURFACES OF THE SECOND DEGREE.

[*Professor Finlay.*]

It is well known that the general equation to surfaces of the second degree can always be reduced to the form

$$u = Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + G = 0 \ldots\ldots\ldots (a);$$

and if the equations of two planes cutting the surface (a) are

$$v = ax + \beta y + \gamma z + 1 = 0,\ v' = a'x + \beta' y + \gamma' z + 1 = 0 \ldots (b),$$

it is clear that the equation to a surface of the second degree passing through the two curves of intersection will be

$$u = mvv' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c),$$

m being an arbitrary constant: for equation (c) is satisfied by the simultaneous equations $u = 0$, $v = 0$,-and $u = 0$, $v' = 0$; and therefore the surface (c) passes through the two plane curves represented by these two systems of equations.

Now since every section of a sphere by a plane is a circle, it is evident that when the surface (c) is a sphere, the planes (b) will become circular sections of the surface (a). But by developing equation (c) we get

$$\begin{aligned} Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + G = {} & m(aa'x^2 + \beta\beta' y^2 + \gamma\gamma' z^2) \\ & + m\{(\beta\gamma' + \beta'\gamma)yz + (\gamma a' + \gamma' a)zx + (a\beta' + a'\beta)xy \\ & + (a + a')x + (\beta + \beta')y + (\gamma + \gamma')z + 1\} \ldots\ldots\ldots\ldots\ldots (c'); \end{aligned}$$

hence (the coordinates being rectangular) the conditions in order that the surface (c) may be a sphere, are

$$\left.\begin{aligned} & maa' - A = m\beta\beta' - B = m\gamma\gamma' - C, \\ & a\beta' + a'\beta = 0,\ \beta\gamma' + \beta'\gamma = 0,\ \gamma a' + \gamma' a = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (d).$$

The last three of these equations give

$$\frac{a}{a'} = -\frac{\beta}{\beta'},\ \frac{a}{a'} = -\frac{\gamma}{\gamma'},\ \frac{\beta}{\beta'} = -\frac{\gamma}{\gamma'};$$

which are evidently inconsistent with each other when the quantities $a, \beta, \gamma,\ a', \beta', \gamma'$ are all finite. If however we assume $a = 0$, $a' = 0$, equations (d) are reduced to

$$\frac{\gamma}{\beta} = -\frac{\gamma'}{\beta'},\ m\beta\beta' = B - A,\ m\gamma\gamma' = C - A;$$

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from which we easily deduce

$$\frac{\gamma\gamma'}{\beta\beta'} = \frac{C-A}{B-A} = -\frac{\gamma^2}{\beta^2} = -\frac{\gamma'^2}{\beta'^2};$$

$$\therefore \frac{\gamma}{\beta} = \left(\frac{C-A}{A-B}\right)^{\frac{1}{2}}, \quad \frac{\gamma'}{\beta'} = -\left(\frac{C-A}{A-B}\right)^{\frac{1}{2}},$$

and equations (b) become

$$y + z\left(\frac{C-A}{A-B}\right)^{\frac{1}{2}} + \frac{1}{\beta} = 0, \quad y - z\left(\frac{C-A}{A-B}\right)^{\frac{1}{2}} + \frac{1}{\beta'} = 0 \ldots . (b_1)$$

where β and β' remain indeterminate.

Again, if we assume $\beta = 0, \beta' = 0$; equations (d) are reduced to

$$\frac{a}{\gamma} = -\frac{a'}{\gamma'}, \; maa' = A - B, \; m\gamma\gamma' = C - B,$$

$$\text{hence, } \frac{aa'}{\gamma\gamma'} = \frac{A-B}{C-B} = -\frac{a^2}{\gamma^2} = -\frac{a'^2}{\gamma'^2};$$

$$\frac{a}{\gamma} = \left(\frac{A-B}{B-C}\right)^{\frac{1}{2}}, \quad \frac{a'}{\gamma'} = -\left(\frac{A-B}{B-C}\right)^{\frac{1}{2}};$$

and by substituting these expressions in equations (b) we obtain

$$x\left(\frac{A-B}{B-C}\right)^{\frac{1}{2}} + z + \frac{1}{\gamma} = 0, \quad -x\left(\frac{A-B}{B-C}\right)^{\frac{1}{2}} + z + \frac{1}{\gamma'} = 0 \ldots . (b_2)$$

where γ and γ' are indeterminate.

If we assume $\gamma = 0, \gamma' = 0$; equations (d) are reduced to

$$\frac{\beta}{a} = -\frac{\beta'}{a'}, \; m\beta\beta' = B - C, \; maa' = A - C;$$

$$\therefore \frac{\beta\beta'}{aa'} = \frac{B-C}{A-C} = -\frac{\beta^2}{a^2} = -\frac{\beta'^2}{a'^2},$$

$$\frac{\beta}{a} = \left(\frac{B-C}{C-A}\right)^{\frac{1}{2}}, \quad \frac{\beta'}{a'} = -\left(\frac{B-C}{C-A}\right)^{\frac{1}{2}};$$

and by these substitutions equations (b) give

$$x + y\left(\frac{B-C}{C-A}\right)^{\frac{1}{2}} + \frac{1}{a} = 0, \; x - y\left(\frac{B-C}{C-A}\right)^{\frac{1}{2}} + \frac{1}{a'} = 0 \ldots . . (b_3)$$

a and a' being indeterminate.

Equations (b_1), (b_2), and (b_3) show that in general *every surface of the second degree has six series of circular sections (real or imaginary) parallel to the three principal axes of the surface.* The position of the planes of the circular sections is obviously independent of the coefficients D, E, F, and the absolute term G of equation (a).

Case I. When the given surface is an ellipsoid, by taking its principal axes as axes of coordinates equation (a) becomes

$$Ax^2 + By^2 + Cz^2 = 1;$$

hence A, B, C are positive; and if $A < B < C$, equations (b_2) are real, but (b_1) and (b_3) are imaginary: consequently the equations

$$x\left(\frac{A-B}{B-C}\right)^{\frac{1}{2}} + z + \frac{1}{\gamma} = 0, \quad -x\left(\frac{A-B}{B-C}\right)^{\frac{1}{2}} + z + \frac{1}{\gamma'} = 0,$$

indicate two series of real circular sections, parallel to the axis of y, or to the mean axis of the ellipsoid.

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Case II. When the surface (a) is a hyperboloid of one sheet, by referring the surface to its principal axes, equation (a) becomes

$$Ax^2 + By^2 - Cz^2 = 1;$$

hence C is negative, and if A < B equations (b_1) give

$$y + z\left(\frac{C+A}{B-A}\right)^{\frac{1}{2}} + \frac{1}{\beta} = 0, \quad y - z\left(\frac{C+A}{B-A}\right)^{\frac{1}{2}} + \frac{1}{\beta'} = 0,$$

but equations (b_2) and (b_3) give imaginary results. Hence the hyperboloid of one sheet has only two series of real circular sections, which are both parallel to the greater axis of the principal elliptic section, or to the greater real axis of the surface.

Case III. When the surface (a) is a hyperboloid of two sheets, its equation may be reduced to the form

$$Ax^2 - By^2 - Cz^2 = 1;$$

hence if B < C equations (b_2) give the real results

$$x\left(\frac{A+B}{C-B}\right)^{\frac{1}{2}} + z + \frac{1}{\gamma} = 0, \quad -x\left(\frac{A+B}{C-B}\right)^{\frac{1}{2}} + z + \frac{1}{\gamma'} = 0;$$

but the results of (b_1) and (b_3) are imaginary. Hence this surface has only two series of circular sections, the planes of which are both parallel to the axis of y, or the greater imaginary axis of the surface.

Case IV. When the given surface is an elliptic paraboloid, by taking the axis of the surface as the axis of x, and the rectangular diametral planes passing through it as the planes of xz and xy, equation (a) becomes

$$By^2 + Cz^2 - x = 0;$$

hence A = 0, and if B < C equations (b_2) give

$$x\left(\frac{B}{C-B}\right)^{\frac{1}{2}} + z + \frac{1}{\gamma} = 0, \quad -x\left(\frac{B}{C-B}\right)^{\frac{1}{2}} + z + \frac{1}{\gamma'} = 0,$$

but (b_1) and (b_3) are imaginary. Hence the elliptic paraboloid has two series of circular sections, both parallel to the axis of y, which lies in the plane of the principal parabolic section whose parameter is the greater.

Case V. When the surface is a hyperbolic paraboloid, by taking the axes as in the last case, the equation (a) becomes

$$By^2 - Cz^2 - x = 0,$$

hence equations (b_1) give

$$y + z\left(\frac{C}{B}\right)^{\frac{1}{2}} + \frac{1}{\beta} = 0, \quad y - z\left(\frac{C}{B}\right)^{\frac{1}{2}} + \frac{1}{\beta'} = 0,$$

but (b_2) and (b_3) are imaginary. These equations indicate two series of circular sections parallel to the principal axis of the surface. In this case however each of the circles degenerates into a straight line. For by eliminating y and z successively from the equations

$$By^2 - Cz^2 - x = 0, \text{ and } y + z\left(\frac{C}{B}\right)^{\frac{1}{2}} + \frac{1}{\beta} = 0 \ldots\ldots\ldots\ldots (A)$$

we obtain $x = \frac{2z}{\beta}\sqrt{BC} + \frac{B}{\beta^2}$, $x = -\frac{2B}{\beta}y - \frac{B}{\beta^2}$, consequently, the projections of the intersection of the surfaces (A) on the planes of xz and xy are straight lines, and therefore that intersection is a straight line.

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Case VI. When the given surface is a cone, by referring it to its principal axes, the equation becomes

$$Ax^2 + By^2 - Cz^2 = 0,$$

hence when $A < B$ equations (b_1) give

$$y + z\left(\frac{C+A}{B-A}\right)^{\frac{1}{2}} + \frac{1}{\beta} = 0, \quad y - z\left(\frac{C+A}{B-A}\right)^{\frac{1}{2}} + \frac{1}{\beta'} = 0;$$

but (b_2) and (b_3) are imaginary: therefore the cone has two series of circular sections, parallel to the axis of x, or to the greater axis of the surface.

Case VII. When the given surface is an elliptic cylinder, by taking the axis of the cylinder as the axis of z, and the axes of x and y in the two principal diametral planes which pass through the axes, equation (a) becomes

$$Ax^2 + By^2 = 1,$$

hence if $A < B$ equation (b_1) gives

$$y + z\left(\frac{A}{B-A}\right)^{\frac{1}{2}} + \frac{1}{\beta} = 0, \quad y - z\left(\frac{A}{B-A}\right)^{\frac{1}{2}} + \frac{1}{\beta'} = 0,$$

but (b_2) and (b_3) give imaginary results; hence the surface has two series of circular sections parallel to the greater axis of the principal section of the cylinder.

Case VIII. When the given surface is a hyperbolic cylinder, by taking the axes of coordinates as in the last case, equation (a) becomes

$$Ax^2 - By^2 = 1,$$

hence equations (b_3) give

$$x + y\left(\frac{B}{A}\right)^{\frac{1}{2}} + \frac{1}{a} = 0, \quad x - y\left(\frac{B}{A}\right)^{\frac{1}{2}} + \frac{1}{a'} = 0,$$

but (b_1) and (b_2) give imaginary results. These equations appear to indicate two series of circular sections, but it may be shown as in the fifth case, that the circles degenerate into straight lines.

Case IX. When the given surface is a parabolic cylinder, equation (a) may be reduced to the form $By^2 - x = 0$; hence equations (b_1) give $y + \frac{1}{\beta} = 0$. Equations (b_3) give the same result, and (b_2) are imaginary.

Hence it appears that every section by a plane perpendicular to the axis of y is a circle; but the circle in all cases degenerates into a straight line.

From this enumeration of cases it appears that *each of the nine distinct species of surfaces of the second degree (the parabolic cylinder only excepted) has two series of circular sections, but that the circles degenerate into straight lines when the surface is a hyperbolic paraboloid or a hyperbolic cylinder.*

If we assume $a = 0$, $a' = 0$, $\beta = 0$, $\beta' = 0$, equations (d) are reduced to

$$m\gamma\gamma' - C = -B = -A;$$

hence $B = A$ and γ, γ' are arbitrary. In this case therefore *the surface (a) must be a surface of revolution, and the planes (b) of circular section are perpendicular to the axis of revolution.* The same result would obviously be obtained by assuming $\beta = 0$, $\beta' = 0$, $\gamma = 0$, $\gamma' = 0$; or $\gamma = 0$, $\gamma' = 0$, $a = 0$, $a' = 0$.

Lastly, if we assume $a = 0$, $a' = 0$, $\beta = 0$, $\beta' = 0$, $\gamma = 0$, $\gamma' = 0$, equations (d) are reduced to $A = B = C$; hence it is evident that, in this case, *the surface (a) must be a sphere, and the planes (b) of circular section are indeterminate.*

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SURFACES OF THE SECOND ORDER REFERRED TO THREE TANGENT PLANES.

[*Mr. Fenwick.*]

Let

$$ax^2+by^2+cx^2+2dxy+2exz+2fyz+2gz+2hy+2kx+1=0\ldots(1),$$

be the equation of a surface of the second order referred to any axes, and let x, y, z be successively $=0$; then the traces of this surface on the co-ordinate planes will be

$$ax^2+2fyz+by^2+2gz+2hy+1=0\ldots\ldots\ldots\ldots(2)$$
$$ax^2+2exz+cx^2+2gz+2kx+1=0\ldots\ldots\ldots\ldots(3)$$
$$by^2+2dxy+cx^2+2hy+2kx+1=0\ldots\ldots\ldots\ldots(4)$$

Now the coordinate planes will be tangential to (1), when each of the equations (2, 3, 4) denotes a point, a line, or two lines. Moreover, the conditions that (2, 3, 4) may be points or lines are (see any treatise on the conic sections),

$$(fg-ah)^2=(f^2-ab)(g^2-a),$$
$$(eg-ak)^2=(e^2-ac)(g^2-a),$$
$$(dh-bk)^2=(d^2-bc)(h^2-b);$$

or,
$$ah^2+bg^2+f^2-2fgh-ab=0\ldots\ldots\ldots\ldots\ldots(5)$$
$$ak^2+cg^2+e^2-2egk-ac=0\ldots\ldots\ldots\ldots\ldots(6)$$
$$bk^2+ch^2+d^2-2dhk-bc=0\ldots\ldots\ldots\ldots\ldots(7)$$

Eliminate a from (5) and (6), and we have

$$b(gk-e)^2-c(gh-f)^2+f^2k^2-e^2h^2+2ghk\,(eh-fk)=0\ldots(8)$$

Hence eliminating b from (7) and (8), there results the relation

$$(gh-f)^2(c-k^2)^2=\{k(eh+dg)-(de+ghk^2)\}^2,$$

or,
$$(gh-f)\,(c-k^2)=\pm\{k(eh+dg)-(de+ghk^2)\};$$

$$\therefore\ c=\frac{k(eh+dg)-(de+ghk^2)}{gh-f}+k^2=\frac{k(eh+dg-fk)-de}{gh-f}.$$

In a similar way we get

$$a=\frac{g(eh+fk-dg)-ef}{hk-d},\ \text{and}\ b=\frac{h(dg+fk-eh)-df}{gk-e}.$$

The equation therefore in general of a surface of the second order when referred to three tangent planes, is

$$Ax^2+By^2+Cx^2+2dxy+2exz+2fyz+2gz+2hy+2kx+1=0,*$$

of which

$$A=\frac{g(eh+fk-dg)-ef}{hk-d},\ B=\frac{h(dg+fk-eh)-df}{gk-e},\ C=\frac{k(eh+dg-fk)-de}{gh-f}.$$

We may now apply the preceding principles *in finding the locus of the centres of the surfaces of the second order inscribed in a given parallelopiped.*

* I leave it for the student to determine, whether the values of a, b, c, that I have given, on being substituted in (1), make it a *surface of the second order.* If they do not, he must employ the *negative sign* in deducing their values.

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Refer the surfaces to three faces of the parallelopiped as coordinate planes, and let

$$x = \alpha \ldots\ldots\ldots (9), \qquad y = \beta \ldots\ldots\ldots (10), \qquad z = \gamma \ldots\ldots (11),$$

be the equations of the opposite faces: also let (1) be one of the inscribed surfaces. Then the projections on the coordinate planes of the curves which are the intersections, respectively, of (9, 10,), 11 with the surface (1), are

$$az^2+2fyz+by^2+2(e\alpha+g)z+2(d\alpha+h)y+c\alpha^2+2k\alpha+1=0,$$
$$az^2+2exz+cx^2+2(f\beta+g)z+2(d\beta+k)x+b\beta^2+2h\beta+1=0,$$
$$by^2+2dxy+cx^2+2(f\gamma+h)y+2(e\gamma+k)x+a\gamma^2+2g\gamma+1=0.$$

Each of these must denote a point, or a line, in order that (9, 10, 11) may be tangential to (1). Hence we have

$$\{f(e\alpha+g)-a(d\alpha+h)\}^2=(f^2-ab)\{(e\alpha+g)^2-a(c\alpha^2+2k\alpha+1)\},$$
$$\{e(f\beta+g)-a(d\beta+k)\}^2=(e^2-ac)\{(f\beta+g)^2-a(b\beta^2+2h\beta+1)\},$$
$$\{d(f\gamma+h)-b(e\gamma+k)\}^2=(d^2-bc)\{(f\gamma+h)^2-b(a\gamma^2+2g\gamma+1)\}.$$

Developing each of these expressions, reducing, and writing D for $ad^2+be^2+cf^2-2def-abc$, we get the following conditional relations:—

$$D\alpha^2+2\alpha(adh+beg+kf^2-efh-dfg-abk)+ah^2+bg^2+f^2-2fgh-ab=0 \ldots\ldots (12)$$

$$D\beta^2+2\beta(adk+cfg+he^2-efk-deg-ach)+ak^2+cg^2+e^2-2egk-ac=0 \ldots\ldots (13)$$

$$D\gamma^2+2\gamma(cfh+bek+gd^2-dfk-deh-bcg)+bk^2+ch^2+d^2-2dhk-bc=0 \ldots\ldots (14)$$

Take (5) from (12), (6) from (13), and (7) from (14); then we have

$$D\alpha+2(adh+beg+kf^2-efh-dfg-abk)=0 \ldots\ldots (15)$$
$$D\beta+2(adk+cfg+he^2-efk-deg-ach)=0 \ldots\ldots (16)$$
$$D\gamma+2(cfh+bek+gd^2-dfk-deh-bcg)=0 \ldots\ldots (17)$$

Again, X, Y, Z, being the coordinates of the centre of (1), and D having the same value as in (12, 13, 14),

$$DX=abk+dfg+efh-adh-beg-kf^2 \ldots\ldots (18)$$
$$DY=ach+deg+efk-adk-cfg-he^2 \ldots\ldots (19)$$
$$DZ=dfk+deh+bcg-cfh-bek-gd^2 \ldots\ldots (20)$$

By (15, 18), $X=\frac{1}{2}\alpha$; (16, 19), $Y=\frac{1}{2}\beta$; (17, 20), $Z=\frac{1}{2}\gamma$; *all the inscribed surfaces of the parallelopiped have therefore the same centre, which is also the centre of the circumscribed surfaces.* (*Mathematician*, Vol. II. p. 122.) Hence this property, which is I think new:—*If surfaces of the second order be inscribed in a given parallelopiped, and also others be circumscribed about the same, all these inscribed and circumscribed surfaces will have the same centre.*

R. M. Academy, April 8, 1846.

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PROPERTIES OF THE ELLIPSE.

[*Mr. Thomas Weddle, Newcastle on Tyne.*]

CONJUGATE DIAMETERS. — DIVISION OF AN ELLIPSE INTO EQUAL SECTORS. — MAXIMUM INSCRIBED, AND MINIMUM CIRCUMSCRIBED POLYGONS.

Upon AB, the major axis of an ellipse ACBD, whose centre is O, describe a circle (the figure is easily conceived), and from any point P in the ellipse draw PM perpendicular to AB meeting the circle in Q. Let $AO = a$, $CO = b$, and $\prec QOB = \phi$, then will $x = OM = a\cos\phi$, and $y = PM = \frac{b}{a} QM = b\sin\phi$, hence the equation to the ellipse may be put under the simple form

$$x = a\cos\phi, \quad y = b\sin\phi \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The introduction of the eccentric anomaly (as ϕ is called in astronomy) into the equation of the ellipse, will frequently be found advantageous; and it is one of the objects of this paper to show how greatly it simplifies the investigation of many properties of this important curve.

Throughout this paper the point Q_m in the circle will correspond to P_m in the ellipse, and ϕ_m will denote the value of ϕ at these points.

The equation to the chord $P_1 P_2$ is therefore

$$\frac{y - b\sin\phi_1}{b(\sin\phi_1 - \sin\phi_2)} = \frac{x - a\cos\phi_1}{a(\cos\phi_1 - \cos\phi_2)},$$

which readily reduces to

$$\frac{x\cos\frac{1}{2}(\phi_1 + \phi_2)}{a} + \frac{y\sin\frac{1}{2}(\phi_1 + \phi_2)}{b} = \cos\tfrac{1}{2}(\phi_2 - \phi_1) \ldots . (2)$$

This equation, which is that of the chord $P_1 P_2$, is very neat.

The length of the chord $P_1 P_2 = \sqrt{\{a^2(\cos\phi_1 - \cos\phi_2)^2 + b^2(\sin\phi_1 - \sin\phi_2)^2\}}$; or, by reduction,

$$P_1 P_2 = 2\sin\tfrac{1}{2}(\phi_2 - \phi_1).\sqrt{\{a^2\sin^2\tfrac{1}{2}(\phi_1 + \phi_2) + b^2\cos^2\tfrac{1}{2}(\phi_1 + \phi_2)\}} \ldots . (3)$$

The coordinates of the middle point of $P_1 P_2$ are $\frac{1}{2}a(\cos\phi_1 + \cos\phi_2)$ and $\frac{1}{2}b(\sin\phi_1 + \sin\phi_2)$, whence the equation to the radius bisecting $P_1 P_2$ is

$$\frac{x}{a(\cos\phi_1 + \cos\phi_2)} = \frac{y}{b(\sin\phi_1 + \sin\phi_2)},$$

or,

$$\frac{x}{a\cos\frac{1}{2}(\phi_1 + \phi_2)} = \frac{y}{b\sin\frac{1}{2}(\phi_1 + \phi_2)} \ldots\ldots\ldots\ldots (4)$$

If P be the point in which the radius bisecting $P_1 P_2$ meets the ellipse, we have (1, 4)

$$\phi = \tfrac{1}{2}(\phi_1 + \phi_2) \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

If $\phi_1 = \phi_2 = \phi$, then the chord $P_1 P_2$ becomes the tangent at P, the equation of which is thus found to be (2),

$$\frac{x\cos\phi}{a} + \frac{y\sin\phi}{b} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

If OP′ be the radius conjugate to OP, and ϕ' the value of ϕ at P′, the equation to OP′ will be

$$\frac{x}{a\cos\phi'} = \frac{y}{b\sin\phi'} \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

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Now, since OP′ and OP are conjugate, (6) and (*a*) are parallel,

$$\therefore \quad \tan\phi' = -\cot\phi = \tan\left(\phi \pm \frac{\pi}{2}\right);$$

$$\therefore \quad \phi' = \phi \pm \frac{\pi}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

This gives two points on the ellipse conjugate to the point P, though only one diameter conjugate to OP.

The equation (7) suggests the following method of finding the diameter conjugate to a given one.

(A). Upon AB the major axis describe a circle; from the extremity P of the given diameter draw PQ perpendicular to AB meeting the circle in Q, and join Q and the centre O; draw Q′O perpendicular to QO meeting the circle in Q′, and Q′P′ perpendicular to AB meeting the ellipse in P′; then will OP′ be conjugate to OP.

If $(x'y')$ be the point conjugate to (x, y) we shall easily get from (7)

$$x' = \mp a\sin\phi = \mp \frac{a}{b}y, \quad y' = \pm b\cos\phi = \pm \frac{b}{a}x \ldots\ldots(8)$$

Let r be the radius OP, and r' its conjugate OP′; hence we have

$$\left.\begin{aligned} r^2 &= a^2\cos^2\phi + b^2\sin^2\phi \\ r'^2 &= a^2\sin^2\phi + b^2\cos^2\phi \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Adding these two equations, we have

$$r^2 + r'^2 = a^2 + b^2 \ldots\ldots\ldots\ldots\ldots\ldots \quad (10),$$

which is a well known property.

If p be the perpendicular from the centre on the tangent at P, we have (6)

$$p = \frac{ab}{\sqrt{\{a^2\sin^2\phi + b^2\cos^2\phi\}}} = (9)\ \frac{ab}{r'},$$

$$\therefore \ r'p = ab \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

which amounts to another well known property, for it shows that every parallelogram circumscribing an ellipse, and touching it at the extremities of conjugate diameters, is equal to the rectangle under the axes. This theorem is established very simply in a paper in the Camb. Math. Journal, first series, Vol. IV., p. 100, from which the neat property (7) is extracted.*

If r_1' be the radius parallel to the chord P_1P_2, or (which is the same thing) conjugate to that bisecting P_1P_2, we have (3, 5, 9),

$$P_1P_2 = 2r_1' \sin\tfrac{1}{2}(\phi_2 - \phi_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Let P_1, P_2 be two points in the ellipse, and Q_1, Q_2 the corresponding ones in the circle; then will

$$2.\Delta P_1OP_2 = b\sin\phi_2\, a\cos\phi_1 - a\cos\phi_2\, b\sin\phi_1$$

or,
$$\Delta P_1OP_2 = \tfrac{1}{2}ab\sin(\phi_2 - \phi_1) \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

Also,
$$\Delta Q_1OQ_2 = \tfrac{1}{2}a^2\sin(\phi_2 - \phi_1)$$

$$\therefore \quad \Delta P_1OP_2 = \frac{b}{a}\Delta Q_1OQ_2 \ldots \ldots\ldots\ldots\ldots\ldots(14)$$

* Upon reading the paper alluded to above, it occurred to me that the equation to the hyperbola may be put under a form somewhat similar to (1); namely

$$x = a\sec\phi, \ y = b\tan\phi \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

and I found a property analogous to (7). I have, however, been anticipated in the discussion of these topics by a writer in the Camb. and Dublin Math. Journal, new series, No. III., to which I beg to refer. I doubt however, whether the equations (*a*) are capable of so many useful applications as the equations (1).

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Now let P_1OP_2 be an elliptic sector, and Q_1OQ_2 the corresponding circular one; if we divide these sectors into corresponding elemental sectors or triangles (as they will be in the limit), and apply (14) to these elements, we have

element of sector $P_1OP_2 = \frac{b}{a}$. corresponding element of sector Q_1OQ_2,

and hence, taking the sum of all the elements, we get

$$\text{sector } P_1OP_2 = \frac{b}{a}. \text{ sector } Q_1OQ_2 \quad \ldots\ldots\ldots\ldots\ldots (15)$$

From (14) and (15), we see that

$$\text{elliptic segment upon } P_1P_2 = \frac{b}{a}. \text{ circular segment upon } Q_1Q_2 \ldots . (16)$$

By (14, 15, 16) we have the following theorem:

(B). If P_1, P_2 be two points in an ellipse whose centre is O, and P_1Q_1, P_2Q_2 perpendiculars to the major axis meeting the circle described on this axis in Q_1 and Q_2 respectively, the ratios of the triangle P_1OP_2 to the triangle Q_1OQ_2, of the sector P_1OP_2 to the sector Q_1OQ_2, and of the segment upon P_1P_2 to the segment upon Q_1Q_2, are each equal to the ratio of the minor axis to the major.

Since sector $Q_1OQ_2 = \frac{1}{2}a^2(\phi_2 - \phi_1)$, we have (15)

$$\text{sector } P_1OP_2 = \tfrac{1}{2}ab(\phi_2 - \phi_1) \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

If $\phi_1 = 0$ and $\phi_2 = 2\pi$, the sector becomes the whole ellipse, and we have the usual formula (πab) for its area.

From (13, 17),

$$\text{segment upon } P_1P_2 = \tfrac{1}{2}ab\{(\phi_2 - \phi_1) - \sin(\phi_2 - \phi_1)\} \ldots\ldots\ldots (18)$$

Let $Q_1, Q_2 \ldots Q_n$ be the angles of a regular polygon of n sides, inscribed in the circle, and $P_1, P_2 \ldots P_n$ the points in which the perpendiculars on the major axis from $Q_1, Q_2 \ldots Q_n$ intersect the ellipse. Then will $\phi_{m+1} - \phi_m = \frac{2\pi}{n}$, and hence (17)

$$\text{sector } P_1OP_2 = \text{sector } P_2OP_3 = \ldots = \text{sector } P_{n-1}OP_n = \text{sector } P_nOP_1 = \frac{\pi ab}{n} \quad \ldots . (19)$$

From which we see that

(C). Whenever n is such that a regular polygon of n sides can be described geometrically, an ellipse can be divided geometrically into n equal sectors, in an indefinite number of ways.

The following construction gives one of these ways:

(D). Upon AB, the major axis of the ellipse, describe a circle, in which inscribe a regular polygon $Q_1 Q_2 \ldots Q_n$ of the proposed number (n) of sides; from $Q_1, Q_2 \ldots Q_n$ demit perpendiculars upon AB meeting the ellipse in $P_1, P_2 \ldots P_n$, and join $P_1, P_2 \ldots P_n$ and the centre O; the ellipse is divided into n equal sectors P_1OP_2, $P_2OP_3 \ldots P_{n-1}OP_n$, P_nOP_1.

Since $\angle Q_1OB$ or ϕ_1 may obviously have any value between 0 and $\frac{2\pi}{n}$, the number of ways in which this division can be effected is indefinite, as has been previously remarked (C).

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We have evidently,

$$\phi_2=\phi_1+\frac{2\pi}{n}\ldots\ldots\phi_m=\phi_1+\frac{2(m-1)\pi}{n}\ldots\ldots\phi_n=\phi_1+\frac{2(n-1)\pi}{n}\ldots\ldots(20)$$

Scholium. (D) contains the solution of exercise 87, proposed by me at p. 172. The particular case in which $n=5$, has been discussed by Mr. Dalmahoy in No. 2, Vol. II., of this work; the general construction here given was however discovered independently, indeed it was communicated to the Editors of the Mathematician some time before the publication of Mr. Dalmahoy's paper. The above solution is remarkable both on account of its simplicity and generality; moreover as will be seen below, the same process determines the angular points of a maximum inscribed, and the points of contact of a minimum circumscribed polygon of the given number of sides.

I may also observe, that could we divide a circular arc into n equal parts, we could divide an elliptic sector into n equal sectors, (see (Γ) below); and hence we can always bisect an elliptic sector.*

Let $P_1, P_2 \ldots P_n$ be the angles of any polygon inscribed in the ellipse, and $Q_1, Q_2 \ldots\ldots Q_n$ the corresponding points in the circle; from (14) we have

$$\Delta P_1OP_2 = \frac{b}{a}.\ \Delta Q_1OQ_2$$

$$\Delta P_2OP_3 = \frac{b}{a}.\ \Delta Q_2OQ_3$$

$$\vdots$$

$$\Delta P_{n-1}OP_n = \frac{b}{a}.\ \Delta Q_{n-1}OQ_n$$

$$\Delta P_nOP_1 = \frac{b}{a}.\ \Delta Q_nOQ_1.$$

By addition, polygon $P_1P_2\ldots P_n = \frac{b}{a}.$ polygon $Q_1Q_2..Q_n$ .....(21)

Hence the polygon $P_1P_2\ldots P_n$ will be a maximum, when the polygon $Q_1Q_2\ldots Q_n$ is so; now, (the number of sides being constant,) the latter is a maximum when it is regular, its area then being $= \frac{1}{2}a^2n \sin\frac{2\pi}{n}$,

$$\therefore \text{ polygon } P_1P_2\ldots P_n \text{ (when a maximum)} = \tfrac{1}{2}abn \sin\frac{2\pi}{n}\ldots\ldots(22)$$

Since, as before, the angle Q_1OB or ϕ_1 may have any value between 0 and $\frac{2\pi}{n}$ we infer that

(E). The number of sides being given, an indefinite number of maximum polygons may be inscribed in an ellipse.

The determination of the angular points $P_1, P_2\ldots\ldots P_n$ of such a polygon is to be effected as in (D), and the relation among $\phi_1, \phi_2\ldots\ldots\phi_n$ is evidently the same here as in (20).

* The division of an ellipsoid, and of some other solids into equal volumes, may be effected by (D). This will be shown in a subsequent paper.

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Since $\phi_{m+1} + \phi_{m-1} = 2\phi_m$, it follows from (2, 6) that

(F). If $P_1 P_2 \ldots P_n$ be a maximum polygon of n sides inscribed in an ellipse, the tangent at any angle P_m is parallel to the chord $P_{m-1} P_{m+1}$.

Conversely,

(G). If a polygon $P_1 P_2 \ldots P_n$ inscribed in an ellipse be such that the tangent at P_m (for all values of m) is parallel to the chord $P_{m-1} P_{m+1}$, the polygon is a maximum one inscribed in the ellipse.

For (2, 6), the condition of parallelism is $\tan\phi_m = \tan\frac{1}{2}(\phi_{m-1}+\phi_{m+1})$, the solutions of which are contained in $p\pi+\phi_m = \frac{1}{2}(\phi_{m-1} + \phi_{m+1})$, or $\phi_{m-1}+\phi_{m+1} = 2\phi_m + 2p\pi$, where p is any positive or negative integer; now $\phi_{m-1} < \phi_m < \phi_{m+1}$, and each > 0 but $< 2\pi$, hence p must $= 0$, $\therefore \phi_{m-1}+\phi_{m+1} = 2\phi_m$, the same as before.

Let r'_m denote the radius parallel to the side $P_m P_{m+1}$. Since $\phi_{m+1} - \phi_m = (20)\ \frac{2\pi}{n}$, we have (12)

$$P_m P_{m+1} = 2 \sin\frac{\pi}{n} . r'_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

Now when the polygon is a hexagon, $2\sin\frac{\pi}{n} = 2\sin\frac{\pi}{6} = 1$, and, in this case, $P_m P_{m+1} = r'_m$; hence,

(H). If a maximum hexagon be inscribed in an ellipse, each side is equal to the radius parallel to it.

The analogy between this theorem and Euclid 15, IV. Cor., is too obvious to escape observation.

Again (3, 20)

$$P_m P^2_{m+1} = 4\sin^2\frac{\pi}{n} . \left\{a^2 \sin^2\left(\phi_1 + (2m-1)\frac{\pi}{n}\right) + b^2\cos^2\left(\phi_1 + (2m-1)\frac{\pi}{n}\right)\right\}$$

$$= 2\sin^2\frac{\pi}{n} . \left\{a^2 + b^2 - (a^2 - b^2)\cos\left(2\phi_1 + (2m-1)\frac{2\pi}{n}\right)\right\} \ldots . (24)$$

Take $m = 1, 2 \ldots\ldots n$ successively, and add, recollecting that

$$\cos\left(2\phi_1 + \frac{2\pi}{n}\right) + \ldots . + \cos\left\{2\phi_1 + (2n-1)\frac{2\pi}{n}\right\} = \frac{\cos(2\phi_1 + 2\pi).\sin 2\pi}{\sin\frac{2\pi}{n}} = 0 ;$$

$$\therefore P_1P_2^2 + P_2P_3^2 + \ldots . + P_{n-1}P_n^2 + P_n P_1^2 = 2n\sin^2\frac{\pi}{n} . (a^2 + b^2) \ldots \quad (25)$$

Since,

$$OP^2_m = a^2\cos^2\phi_m + b^2\sin^2\phi_m = \tfrac{1}{2} . \left\{a^2 + b^2 + (a^2 - b^2)\cos\left(2\phi_1 + (m-1)\frac{4\pi}{n}\right)\right\},$$

we get, in like manner,

$$OP_1^2 + OP_2^2 + \ldots . + OP_n^2 = \tfrac{1}{2} n(a^2 + b^2) \ldots\ldots\ldots\ldots (26)$$

Let S or (u, v, w) be any point, either in the plane of the ellipse or not, then will

$$SP^2_m = (u - a\cos\phi_m)^2 + (v - b\sin\phi_m)^2 + w^2$$

$$= u^2 + v^2 + w^2 + a^2\cos^2\phi_m + b^2\sin^2\phi_m - 2au\cos\phi_m - 2bv\sin\phi_m$$

$$= SO^2 + \tfrac{1}{2}(a^2 + b^2) + \tfrac{1}{2}(a^2 - b^2)\cos 2\phi_m - 2au\cos\phi_m - 2bv\sin\phi_m .$$

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If we substitute for ϕ_m from (20), and sum as before, we shall find

$$SP_1^2+SP_2^2+\ldots+PS_n^2 = n.SO^2+\tfrac{1}{2}n(a^2+b^2) \quad \ldots\ldots(27)$$

which evidently includes (26).

(27) may be enunciated as follows:

(I.) If from any point in space lines be drawn to the angular points of a maximum polygon of n sides inscribed in an ellipse, the sum of the squares of these lines is equal to n times a quantity which is got by adding half the sum of the squares of the semiaxes to the square of the distance of the point from the centre of the ellipse.

Instead of $\frac{1}{2}(a^2+b^2)$ in the above expressions, we might substitute the square of one of the equal conjugate radii.

Again, let the tangents at P_1 and P_2 intersect in R_1, and those at Q_1 and Q_2 in S_1. Hence, by (6), if (X_1Y_1) denote R_1, we shall have

$$X_1 = \frac{a\cos\frac{1}{2}(\phi_1+\phi_2)}{\cos\frac{1}{2}(\phi_2-\phi_1)},\quad Y_1 = \frac{b\sin\frac{1}{2}(\phi_1+\phi_2)}{\cos\frac{1}{2}(\phi_2-\phi_1)} \quad \ldots\ldots(28)$$

The equation of OR_1 is therefore

$$\frac{x}{a\cos\frac{1}{2}(\phi_1+\phi_2)} = \frac{y}{b\sin\frac{1}{2}(\phi_1+\phi_2)} \quad \ldots\ldots(29)$$

Now (29) coincides with (4), the equation to the radius bisecting P_1P_2, and we hence deduce the following well known theorem.

(K.) If from the intersection of two tangents to an ellipse, a line be drawn to the centre, it will bisect the chord joining the points of contact.

By (28) we have

$$OR_1 = \frac{\sqrt{\{a^2\cos^2\frac{1}{2}(\phi_1+\phi_2)+b^2\sin^2\frac{1}{2}(\phi_1+\phi_2)\}}}{\cos\frac{1}{2}(\phi_2-\phi_1)} \quad \ldots\ldots(30)$$

Let OR_1 meet the ellipse in T_1; then since the coordinates of T_1 are evidently (29), $a\cos\frac{1}{2}(\phi_1+\phi_2)$ and $b\sin\frac{1}{2}(\phi_1+\phi_2)$, we get

$$OT_1^2 = a^2\cos^2\tfrac{1}{2}(\phi_1+\phi_2)+b^2\sin^2\tfrac{1}{2}(\phi_1+\phi_2)$$

$$\therefore\quad OT_1 = OR_1.\ \cos\tfrac{1}{2}(\phi_2-\phi_1) \quad \ldots\ldots(31)$$

If we denote P_1 by (x_1y_1), we shall obtain

$$2\Delta P_1OR_1 = x_1Y_1-y_1X_1 = (1,28)\ \frac{ab.\{\sin\frac{1}{2}(\phi_1+\phi_2).\cos\phi_1-\cos\frac{1}{2}(\phi_1+\phi_2).\sin\phi_1\}}{\cos\frac{1}{2}(\phi_2-\phi_1)}$$

$$= ab\tan\tfrac{1}{2}(\phi_2-\phi_1).$$

Similarly, $$2\Delta P_2OR_1 = ab\tan\tfrac{1}{2}(\phi_2-\phi_1)$$

$$\therefore\ \Delta P_1OR_1 = \Delta P_2OR_1 = \tfrac{1}{2}ab\tan\tfrac{1}{2}(\phi_2-\phi_1) \quad \ldots\ldots(32)$$

Now $$\Delta Q_1OS_1 = \Delta Q_2OS_1 = \tfrac{1}{2}a^2\tan\tfrac{1}{2}(\phi_2-\phi_1)$$

$$\therefore\ \Delta P_1OR_1 = \Delta P_2OR_1 = \frac{b}{a}.\ \Delta Q_1OS_1 = \frac{b}{a}.\ \Delta Q_2OS_1 \quad \ldots\ldots(33)$$

From (32) we deduce the following theorem, which also follows from (K.)

(L.) If a quadrilateral be formed by two radii of an ellipse and the tangents at their extremities, it will be bisected by the diagonal drawn from the intersections of the tangents to the centre.

From (32), $$\text{quad. } P_1OP_2R_1 = ab.\tan\tfrac{1}{2}(\phi_2-\phi_1) \quad \ldots\ldots(34)$$

also from (33) $$\text{quad. } P_1OP_2R_1 = \frac{b}{a}.\ \text{quad. } Q_1OQ_2S_1 \quad \ldots\ldots(35)$$

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Let $R_1 R_2 \ldots R_n$ be a polygon circumscribing the ellipse, its sides $R_n R_1$, $R_1 R_2 \ldots R_{n-1} R_n$ touching in the points $P_1, P_2 \ldots P_n$; let also $Q_1, Q_2 \ldots Q_n$ be the points in the circle corresponding to $P_1, P_2 \ldots P_n$, and $S_1 S_2 \ldots S_n$ the polygon circumscribing and touching it at $Q_1, Q_2 \ldots Q_n$.

By (35) we have
$$\text{quad.} P_1 O P_2 R_1 = \frac{b}{a} \cdot \text{quad.} Q_1 O Q_2 S_1$$
$$\text{quad.} P_2 O P_3 R_2 = \frac{b}{a} \cdot \text{quad.} Q_2 O Q_3 S_2$$
$$\vdots$$
$$\text{quad.} P_{n-1} O P_n R_{n-1} = \frac{b}{a} \cdot \text{quad.} Q_{n-1} O Q_n S_{n-1}$$
$$\text{quad.} P_n O P_1 R_n = \frac{b}{a} \cdot \text{quad.} Q_n O Q_1 S_n .$$

By addition, $$\text{polygon } R_1 R_2 \ldots R_n = \frac{b}{a} \cdot \text{polygon } S_1 S_2 \ldots S_n \ldots \quad (36)$$

Hence we easily infer that the polygon $R_1 R_2 \ldots R_n$ is a minimum when the polygon $S_1 S_2 \ldots S_n$ is regular, the area of the latter polygon then being $= a^2 n \tan \frac{\pi}{n}$,

$$\therefore \text{polygon } R_1 R_2 \ldots R_n \text{ (when a minimum)} = abn \tan \frac{\pi}{n} \ldots\ldots\ldots (37)$$

and the values of $\phi_2, \phi_3 \ldots \phi_n$ are obviously the same here as in (20). Hence

(M). Having found $P_1, P_2 \ldots P_n$ as in (D), tangents at these points will by their intersection form a minimum polygon $R_1 R_2 \ldots R_n$ of n sides, circumscribing the ellipse. The number of such polygons is obviously indefinite.

Moreover, since $\phi_{m+1} - \phi_m = (20) \frac{2\pi}{n}$, a constant quantity, it readily follows from (28) that,

(N). The number (n) of sides being constant, the locus of the angular points of all the minimum polygons circumscribing an ellipse is a concentric similar ellipse whose semiaxes are $a \sec \frac{\pi}{n}$ and $b \sec \frac{\pi}{n}$.

If $\phi'_1, \phi'_2 \ldots \phi'_n$ refer to the points $R_1, R_2 \ldots R_n$ in this ellipse, we have (28),

$$\phi'_1 = \tfrac{1}{2}(\phi_1 + \phi_2) = \phi_1 + \frac{\pi}{n}, \quad \phi'_2 = \tfrac{1}{2}(\phi_2 + \phi_3) = \phi_1 + \frac{3\pi}{n}, \text{ etc.} \ldots (38)$$

or $\phi'_2 = \phi'_1 + \frac{2\pi}{n}$, $\phi'_3 = \phi'_1 + \frac{4\pi}{n}$, *etc.* Hence (20),

(O). The polygon $R_1 R_2 \ldots R_n$ mentioned in (M) is a maximum polygon inscribed in the ellipse or locus mentioned in (N).

It also follows from the preceding propositions, that

(P). The sides of every maximum polygon of n sides inscribed in an ellipse, touch a concentric similar ellipse whose semiaxes are $a \cos \frac{\pi}{n}$ and $b \cos \frac{\pi}{n}$.

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Let OR_m intersect the ellipse in T_m, then (20, 31)

$$OT_m = OR_m \cdot \cos\frac{\pi}{n} \quad \ldots\ldots \quad \ldots\ldots\ldots\ldots (39)$$

If $n = 3$, we have $OR_m = 2.OT_m$, whence

(Q). If a minimum triangle circumscribe an ellipse, the straight lines drawn from the centre to the angles are bisected by the ellipse.

If r'_m and ρ'_m denote, in the given ellipse and in that mentioned in (N), the radii parallel to $R_m R_{m+1}$, we shall evidently have by (N) and (23), $\rho'_m = r'_m \sec\frac{\pi}{n}$, and $R_m R_{m+1} = 2\sin\frac{\pi}{n}.\rho'_m$.

$$\therefore\ R_m R_{m+1} = 2\tan\frac{\pi}{n}.r'_m \ldots\ldots\ldots\ldots\ldots (40)$$

When $n = 4$, $\tan\frac{\pi}{n} = 1$, and hence $R_m R_{m+1} = 2r'_m$; this is otherwise obvious, for by comparing (7) and (20), it will be seen that,

(R). A minimum quadrilateral circumscribing an ellipse, touches it at the extremities of conjugate diameters, and its sides are consequently parallel and equal to these diameters; hence also, every minimum quadrilateral is a parallelogram.

Since $\phi_m - \phi_{m-1} = \phi_{m+1} - \phi_m = \frac{2\pi}{n}$, it follows from (32), that $\Delta P_m OR_{m-1} = \frac{1}{2}ab\tan\frac{\pi}{n} = \Delta P_m OR_m$, and these triangles having O for their common vertex are of the same altitude, hence base $R_{m-1}P_m$ = base $R_m P_m$; consequently,

(S). The sides of a minimum polygon circumscribing an ellipse are bisected in the points of contact.

This is however a general as well as a known property.

By (O) and (25) we have

$$R_1R_2^2 + R_2R_3^2 + \ldots + R_{n-1}R_n^2 + R_n R_1^2$$
$$= 2n.\sin^2\frac{\pi}{n}\left\{a^2\sec^2\frac{\pi}{n} + b^2\sec^2\frac{\pi}{n}\right\} = 2n.\tan^2\frac{\pi}{n}\left\{a^2+b^2\right\} \ldots (41)$$

Similarly from (26) and (27), we get

$$OR_1^2 + OR_2^2 + \ldots + OR_n^2 = \tfrac{1}{2}n.\sec^2\frac{\pi}{n}\left\{a^2+b^2\right\} \ldots\ldots\ldots (42)$$

and, $$SR_1^2 + SR_2^2 + \ldots + SR_n^2 = n.SO^2 + \tfrac{1}{2}n.\sec^2\frac{\pi}{n}.\left\{a^2+b^2\right\} \ldots (43),$$

S being as in (27) any point in space.

The polygons have hitherto been inscribed in or described about the whole ellipse, but a process similar to (D) will obtain with reference to a segment or sector merely. Thus,

(T). Let P_1 and P_n be the extremities of a given segment of an ellipse, draw P_1Q_1 and $P_n Q_n$ perpendicular to the major axis AB meeting the circle described on AB in Q_1 and Q_n. Divide the circular arc Q_1Q_n into the equal parts Q_1Q_2, $Q_2Q_3 \ldots Q_{n-1}Q_n$; from $Q_2, Q_3 \ldots Q_{n-1}$ draw perpendiculars to AB meeting the arc of the elliptic segment in $P_2, P_3 \ldots P_{n-1}$; then will $P_1P_2 \ldots P_n$ be a maximum polygon of n sides inscribed in the segment;

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and, if tangents be applied at $P_1, P_2 \ldots P_n$ forming by their intersection the polygon $P_1R_1R_2 \ldots R_{n-1}P_n$; this will be a minimum polygon of $n+1$ sides circumscribing the same: also the sector P_1OP_n will be divided into $n-1$ equal sectors P_1OP_2, $P_2OP_3 \ldots P_{n-1}OP_n$.

As (T) may be established exactly in the manner already employed, the demonstration need not be dwelt upon. It is scarcely necessary to observe that the division of the circular arc into equal parts can rarely be effected *geometrically*.

In this paper we have considered the circle to be described upon the major axis, but the same conclusions may be drawn if it be described upon the other, the only difference being that $2a$ will now be the length of the minor axis, and $2b$ that of the major.

The equations (1) and indeed many of the following ones, are likewise true when AB on which the circle is described is any diameter of the ellipse, providing we draw the circular ordinates perpendicular to AB, but the elliptic ones parallel to the diameter (CD) conjugate to AB; thus from any point P in the ellipse draw PM parallel to CD meeting AB in M, and draw QM perpendicular to AB meeting the circle in Q, then if, as before, $\sphericalangle QOB = \phi$, we have OM or $x = a\cos\phi$, and PM or $y = b\sin\phi$.

ON TANGENT PLANES.

[*Mr. James Reid, Prior's Salford.*]

In the Mathematician, Vol. II. p. 83, a method of finding the tangents of plane curves is given. In this paper the same principle is applied to the investigation of tangent planes.

If p, q and r are any whole positive numbers whose sum does not exceed n, a surface of the n^{th} order is represented by the equation

$$\Sigma(A_{pqr}x^p y^q z^r) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since through any given point in the surface innumerable plane curves may be traced upon the surface, and a rectilineal tangent drawn to every curve, the number of such tangents at the given point is infinite. As we are not at liberty to assume that more than two of the tangents lie in one plane, we shall first determine the plane of two tangents, and then prove that it is the locus of all.

Let the equation to a plane passing through a point (xyz) of the surface be

$$Ax + By + Cz + D = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Remove the origin in (1) and (2) by substituting $x_1 + x$, $y_1 + y$, $z_1 + z$ for x, y, z.

By the binomial theorem

$$(x_1+x)^p = x_1^p + px_1^{p-1}x + \frac{p(p-1)}{1.2}x_1^{p-2}x^2 + \text{etc.}$$

$$(y_1+y)^q = y_1^q + qy_1^{q-1}y + \frac{q(q-1)}{1.2}y_1^{q-2}y^2 + \text{etc.}$$

$$(z_1+z)^r = z_1^r + rz_1^{r-1}z + \frac{r(r-1)}{1.2}z_1^{r-2}z^2 + \text{etc.}$$

Hence,

$$(x_1+x)^p(y_1+y)^q(z_1+z)_r = x_1^p y_1^q z_1^r + px_1^{p-1}y_1^q z_1^r x + qx_1^p y_1^{q-1} z_1^r y + rx_1^p y_1^q z_1^{r-1} z + \text{etc.}$$

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Because (x_1, y_1, z_1) is a point in the curve surface $\Sigma(A_{pqr}x_1y_1z_1)=0$, and in accordance with the last result, (1) becomes

$$\Sigma\left\{A_{pqr}x_1^p y_1^q z_1^r\left(p\frac{x}{x_1}+q\frac{y}{y_1}+r\frac{z}{z_1}+\frac{p(p-1)}{2}\frac{x^2}{x_1^2}+pq\frac{xy}{x_1y_1}+etc.\right)\right\}=0 \quad \ldots\ldots(3)$$

The transformation of (2) gives

$$Ax+By+Cz=0 \quad \ldots\ldots(4)$$

For convenience (3) may be written

$$ax+by+cz+dx^2+exy+fxz+etc=0 \quad \ldots\ldots(5)$$

Putting x, y, and z successively equal to zero in (5) we have the traces of the surface upon the three coordinate planes yz, xz, xy.

On the plane yz .... $by+cz+gy^2+etc.=0$ .....(6)

..... xz .... $ax+cz+fxz+etc.=0$ .....(7)

..... xy .... $ax+by+exy+etc.=0$ .....(8)

Similarly for the traces of the plane (4) we have

On the plane yz .... $By+Cz=0$ ..... (9)

..... xz .... $Ax+Cz=0$ .....(10)

..... xy .... $Ax+By=0$ .....(11)

In order that the straight lines (9) and (10) may be tangents to the curves (6) and (7) we have by the method already referred to

$$\frac{B}{C}=\frac{b}{c}, \quad \frac{A}{C}=\frac{a}{c} \quad \ldots\ldots(12)$$

From these equations we derive the relation

$$\frac{A}{B}=\frac{a}{b},$$

and this is precisely the condition necessary and sufficient in order that the straight line (11) may be a tangent to the curve (8).

Therefore, of the three rectilineal tangents to the curve surface which lie in the three coordinate planes, the plane which contains two contains also the third; and as the position of the coordinate planes is arbitrary, the plane which contains *any* two tangents contains every other that can be drawn at that given point.

Wherefore, (4) having its coefficients determined by (12), the plane which it represents is the locus of the rectilineal tangents to the surface (5) at the point (x_1, y_1, z_1). Applying (12) to (4), the latter becomes

$$ax+by+cz=0 \quad \ldots\ldots(13)$$

Remove the origin of coordinates to the point to which (1) and (2) are referred, by substituting $x-x_1$, $y-y_1$, $z-z_1$ for x, y, z, and restore the values of a, b, and c from (3). We have then for the equation to the tangent plane

$$\Sigma(pA_{pqr}x_1^{p-1}y_1^q z_1^r)(x-x_1)+\Sigma(qA_{pqr}x_1^p y_1^{q-1}z_1^r)(y-y_1)+\Sigma(rA_{pqr}x_1^p y_1^q z_1^{r-1})(z-z_1)=0.$$

This equation fails when for particular values of x_1, y_1, z_1 all the coefficients of the variables vanish. In this case (x_1, y_1, z_1) is a *singular* point, and the tangents do not lie in one plane.

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ON THE THEORY OF COLLINEATION.

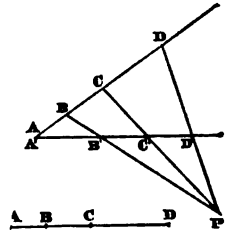
(Continued from page 128.)

[*Mr. Rutherford.*]

VIII.

It may be interesting to shew how the equations expressing the *relation* existing between the coordinates of a point in the one system, and those of a corresponding point in the collinear system are deduced from geometrical considerations.

Let A, B, C, D be four points in a straight line, in one system, and A′, B′, C′ three points in a straight line in the other system. Place the line ABCD making any angle with the line A′B′C′ so that A may coincide with A′. Draw BB′, CC′, and let them be produced and meet in P: join PD intersecting A′B′C′ in D′; then D′ is the point of the latter system which corresponds to the point D of the former system.



For the point D is the point of intersection of the lines ABC and PD in one system, and consequently in conformity with the definition of collinear systems, the point which is to correspond to D must be in the intersection of A′B′C′ and PD, which is therefore the point D′.

From this construction, the relation between the coordinates of a point in one system and those of a corresponding point in the other system may be determined analytically in the following manner.

Let the angular point A or A′ be the origin of coordinates, AD the axis of y, and AD′ that of u; then if

$$A'B' = a, \quad A'C' = a_1, \quad A'D' = u_1$$
$$AB = \beta, \quad AC = \beta_1, \quad AD = y_1,$$

the equations of BB′ and CC′ will be respectively

$$\frac{y}{\beta} + \frac{u}{a} = 1 \ldots\ldots (1); \qquad \frac{y}{\beta_1} + \frac{u}{a_1} = 1 \ldots\ldots\ldots (2)$$

Hence the equation of a line through the intersection of (1) and (2) is

$$\left(\frac{y}{\beta} + \frac{u}{a} - 1\right) + \lambda\left(\frac{y}{\beta_1} + \frac{u}{a_1} - 1\right) = 0 \ldots\ldots\ldots (3),$$

where λ is arbitrary.

If the line (3) passes through D, then $u = 0$, and $y = y_1$; hence

$$\left(\frac{y_1}{\beta} - 1\right) + \lambda\left(\frac{y_1}{\beta_1} - 1\right) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4);$$

and if in (3) we take $y = 0$, and $u = u_1$; then

$$\left(\frac{u_1}{a} - 1\right) + \lambda\left(\frac{u_1}{a_1} - 1\right) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Eliminating λ between the equations (4, 5) we get

$$u_1 = \frac{py_1}{my_1 + 1} \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where $$p = \frac{aa_1(\beta_1 - \beta)}{\beta\beta_1(a_1 - a)}, \text{ and } m = \frac{a\beta_1 - a_1\beta}{\beta\beta_1(a_1 - a)}.$$

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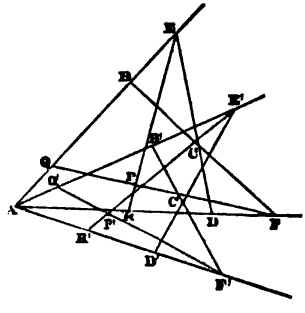
Equation (6) is, in form, identical with that obtained in (III, 4') by the transformation of coordinates, an operation which evidently corresponds to the preceding geometrical construction.

IX.

Let ABCD be any quadrilateral in the system (xy), and A'B'C'D' the corresponding quadrilateral in the system (tu). For simplicity, let the points A and A' be coincident, and let P be a given point in the former system to which a point P' in the latter system is to correspond.

Let AB and CD be produced and meet in E, and A'B', C'D' in E': then E and E' are corresponding points, since they are the intersection of corresponding lines. In a similar manner F and F', the intersections of BC, AD, and of B'C', A'D' are corresponding points. Through P draw the lines EPR and FPQ meeting AE in Q, and AF in R. In the lines AE' and AF' find by the last article (VIII) the points Q' and R' which correspond to the points Q and R: draw F'Q' and E'R' intersecting in P'; then P' will be the point in one system corresponding to the given point P in the other system.



It is unnecessary to give a formal proof of the determination of the point P', inasmuch as the condition of the definition of collinear systems is complied with in every particular.

Let AD, AB be the axes of x and y, and AD', AB' those of t and u, and let $AF = a$, $AE = \beta$. Denote the coordinates of the point P by x_1y_1 and those of the corresponding point P' by t_1u_1; then the equations of the lines EPR and FPQ are respectively

$$(EPR)\ldots\ldots x_1y+(\beta-y_1)x=\beta x_1 \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$(FPQ)\ldots\ldots(a-x_1)y+y_1x=ay_1 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Making $y=0$ in (1) and $x=0$ in (2) we obtain

$$AR=\frac{\beta x_1}{\beta-y_1}, \quad AQ=\frac{ay_1}{a-x_1};$$

hence for the corresponding points R' and Q' we have (VIII, 6)

$$AQ'=\frac{p.AQ}{m.AQ+1}=\frac{pay_1}{may_1-x_1+a} \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$AR'=\frac{q.AR}{n.AR+1}=\frac{q\beta x_1}{n\beta x_1-y_1+\beta} \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Again, since the points E' and F' are the points corresponding to E and F, we get in a similar manner (VIII. 6)

$$AE'=\frac{p\beta}{m\beta+1}, \quad AF'=\frac{qa}{na+1}.$$

Consequently the equations of the lines E'R' and F'Q' are respectively

$$(E'R')\ldots\ldots p(n\beta x_1-y_1+\beta)t+qx_1(m\beta+1)u=pq\beta x_1\ldots\ldots\ldots(5)$$

$$(F'Q')\ldots\ldots py_1(na+1)t+q(may_1-x_1+a)u=pqay_1\ldots\ldots(6),$$

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and the values of the coordinates $(t_1 u_1)$ of the point of intersection P′ of the lines (5, 6) are readily found from these equations, viz.

$$u_1 = \frac{py_1}{my_1^2 + nx_1 + 1}, \qquad t_1 = \frac{qx_1}{my_1 + nx_1 + 1},$$ *

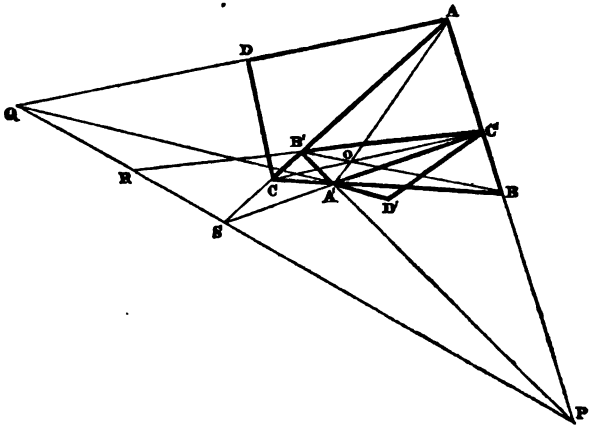
which are the forms obtained in (III, 4). In a similar way it may be shown generally that if xy are the coordinates of a point in one system, the coordinates of its corresponding point in the other system are as given in (II, 3),

$$u = \frac{m'y + n'x + p'}{my + nx + 1}, \quad t = \frac{m''y + n''x + p''}{my + nx + 1}.$$

X.

The *collinear axis* and the *collinear centre* are determined geometrically in the following manner.

Let ABCD and A′B′C′D′ be two collinear and collinearly situated



quadrilaterals: produce AB and A′B′ and let them intersect in the point P: also produce AD and A′D′ and let them intersect in the point Q: draw the line PQ, then PQ is the collinear axis. Also draw the lines AA′, BB′ through the corresponding points A, A′ and B, B′, and let them meet in O; then the point of intersection O is the collinear centre.

From what has been shown in (V), it is evident that each pair of corresponding sides of the collinear quadrilaterals will, when produced if necessary, intersect in the collinear axis PQ, and also that the lines drawn through each pair of corresponding points will pass through O, the collinear centre.

APPLICATION OF THE COLLINEAR THEORY TO TRANSVERSALS.

XI.

We come now to apply the collinear theory to the demonstration of some familiar theorems in transversals.

* In the determination of these values of t_1 and u_1, from the equations (5, 6), the resulting fractional values are reduced to the simple form here given, by expunging the factor $\alpha y_1 + \beta x_1 - \alpha\beta$, which occurs in both terms of the fraction.

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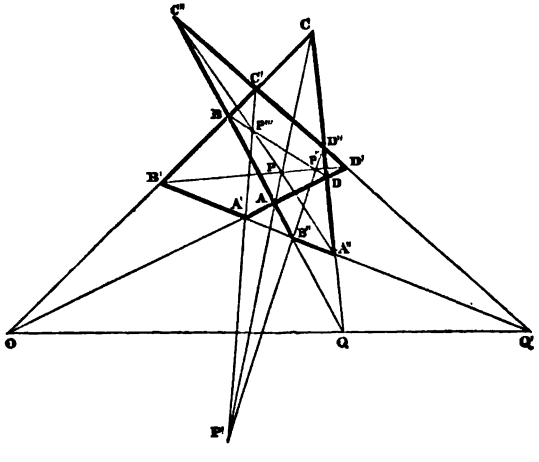
(1.) *If straight lines be drawn from the angles of a triangle through any point in the plane of the triangle to meet the sides, and the lines joining these points of intersection and the sides of the triangle be produced to meet, the three points of intersection will be in the same straight line.*

Let ABC be the triangle (see last figure), and O the point in its plane through which and the angular points of the triangle, the lines AOA′, BOB′, COC′ are drawn. Also draw A′B′, B′C′, C′A′ and let them be produced and meet the sides AB, BC, AC in P, R, S; then will the points P, R, S be in a straight line.

For the point O can be regarded as the collinear centre, and A′, B′, C′ as points corresponding respectively to the points A, B, C: therefore AB, A′B′; BC, B′C′; AC, A′C′ are corresponding lines; and since (v) corresponding lines intersect in the collinear axis, the three points of intersection P, R, S must be situated in the collinear axis PRS, and therefore they are in a straight line.

(2.) *From each of three points in the same straight line draw two straight lines so as to form, by their mutual intersections, three quadrilaterals; the diagonals of these quadrilaterals, three and three, (viz. one diagonal of each quadrilateral), will pass through the same point.*

Let O, Q, Q′ be three points in one straight line, and from each of these points let two lines be drawn so that, by their mutual intersection, the four quadrilaterals ABCD, A′B′C′D′, A″B″B″D″ may be formed; then shall the diagonals AC, A′C′, B″D″ pass through the same point P′.



For the lines joining Q,Q′; A,A′ and D,D′ pass through the same point O; and therefore to the points Q,Q′ the points A,A′ as well as the points D,D′ may be taken as corresponding points; hence the lines QA, Q′A′, are corresponding lines, and also the lines QD, Q′D′. Consequently the points B, B′ and C, C′ which are situated in the intersection of OBC with those lines, may again be taken as corresponding points; and therefore AC, A′C′ and BD, B′D′ are corresponding lines; but since corresponding lines of collinear and collinearly situated figures, intersect in the collinear axis (v); therefore the corresponding lines QA, Q′A′; QD, Q′D′; AC, A′C′; BD, B′D′ must all intersect in one line, the collinear axis of the systems ABCD and A′B′C′D′. But QA, Q′A′ intersect in B″, and QD, Q′D′ in D″; hence the two corresponding lines AC, A′C′, as well as the two other

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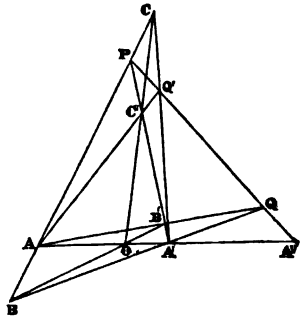
corresponding lines BD, B′D′ must intersect in the collinear axis B″D″; therefore the diagonals AC, A′C′, B″D″ tend to the same point P′ in the collinear axis B″D″. Again since B, B′ and D, D′ are corresponding points, the corresponding lines BD and B′D′ must intersect also in the collinear axis B″D″. In a similar manner it may be shown that the diagonals AC, BD, B′D′ pass through one point, and the diagonals A′C′, A″C″, BD pass through one point: therefore the truth of the theorem is established.

We have therefore determined *four* new points in which three straight lines constantly intersect, and therefore these points may be taken as collinear centres, and we may determine other lines which intersect three and three in one point.

Hence from the last theorem it follows that *if the three collinear centres of three collinear figures, which are two and two collinearly situated, are in the same straight line, the three collinear axes intersect in the same point.*

(3.) *Three points are situated in a straight line, and lines are drawn from these points to any arbitrary point: from two of the given points draw lines to any point in the line passing through the arbitrary point and the third given point, intersecting the two lines drawn through the arbitrary point and the two specified points; then the line drawn through the two points of intersection will always pass through a fixed point in the straight line in which the three given points are situated.*

Let A, A′, A″ be three points in a straight line, and P any arbitrary point; draw the lines PA, PA′, PA″, and let Q be any point in the line PA″. Let the lines drawn through Q and the two points A, A′ intersect PA and PA′ in B and B′; then the line joining B, B′ will pass through a fixed point O in the line AA′A″.



For A, A′ and B, B′ can be regarded as corresponding points of two systems, and the point of intersection (O) of the lines joining these corresponding points as the collinear centre. Now since A and A′ are on opposite sides of the point O, the principle in (VII) will here be applicable; therefore AB′, A′B are corresponding lines as well as AB, A′B′, and they must therefore intersect in the collinear axis; PQA″ is consequently the collinear axis. Take any point C in PA and draw OC intersecting PA′ in C′; then C and C′ are corresponding points, and hence AC′ and A′C are corresponding lines; these lines must then intersect in a point Q′ in the collinear axis, and therefore, conversely, if from any arbitrary point Q in the line PA″ two lines be drawn through A and A′, meeting PA in B and PA′ in B′, the line joining the points of intersection B and B′ will always pass through a fixed point O in AA′, which is the collinear centre of the two systems.

In this manner a variety of propositions may be readily demonstrated, and it will be seen that the collinear theory applies very happily to a large class of interesting and elegant theorems.

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APPLICATION TO LINES OF THE SECOND DEGREE.

XII.

If straight lines be drawn through a fixed point, and intersecting a line of the second degree, each of these lines will in general cut the curve in two points; then each pair of intersections of one of the straight lines can be regarded as corresponding points of the same collinear and collinearly situated system, and the fixed point as the collinear centre.

Let $u = au^2+2btu+ct^2+2du+2et+f=0$ .....(1)

be the equation of a line of the second degree, the fixed point being the origin of coordinates, and any two straight lines drawn through it as coordinate axes. In (1) write for t and u the values given in (III) eq. 4, viz.

$$u=\frac{py}{my+nx+1}, \quad t=\frac{px}{my+nx+1};$$

then we have the equation

$$(ap^2+2dmp+fm^2)y^2+2(bp^2+dnp+emp+fmn)xy+(cp^2+2enp+fn^2)x^2 +2(dp+fm)y+2(ep+fn)x+f=0 \ldots\ldots\ldots\ldots(2)$$

Hence the points, xy, corresponding to the points, tu, in the curve (1), are situated in a curve of the second degree (2); and if they are restricted to the curve (1), the equations (1) and (2) must be identical. That this may be the case, we must have the following five conditions, viz.

$$ap^2+2dmp+fm^2=a, \qquad dp+fm=d,$$
$$bp^2+dnp+emp+fmn=b, \qquad ep+fn=e.$$
$$cp^2+2enp+fn^2=c,$$

Now these five equations can be satisfied either by the values

$$m=0, \quad n=0, \quad p=1,$$

or,
$$m=\frac{2d}{f}, \quad n=\frac{2e}{f}, \quad p=-1.$$

The former system of values gives $u=y$, and $t=x$, and the points xy and tu are coincident. The latter set of values gives

$$u=\frac{-y}{2\frac{d}{f}y+2\frac{e}{f}x+1}, \quad t=\frac{-x}{2\frac{d}{f}y+2\frac{e}{f}x+1} \ldots\ldots\ldots(3)$$

Hence we infer that not only can the two points of intersection be regarded as corresponding points, but also because $p=-1$, each pair of corresponding points can be mutually interchanged (VII).

The equation of the collinear axis, which is (V, 7)

$$my+nx+1-p=0,$$

becomes in this case by (3)

$$dy+ex+f=0 \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

which is also the equation of the polar of the origin. The equation of the counter axis (the same for both systems) is (VI, 8)

$$dy+ex+\tfrac{1}{2}f=0 \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

which shows that the counter axis is parallel to the collinear axis, and bisects the distance of the collinear centre from the collinear axis.

Hence it follows that if from any point P two tangents PA and PB be

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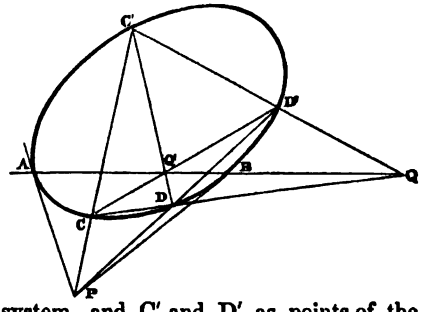
drawn to a line of the second degree, the chord of contact AB may be considered as the collinear axis of two collinear and collinearly situated systems, and the point P as the collinear centre.



For let the lines PCC′ and PDD′ be drawn cutting the curve in the points C, C′ and D, D′; then not only can C, C′ and D, D′ be regarded as corresponding points, but each pair of corresponding points can be mutually interchanged; and therefore not only can C and D be regarded as points of the one system, and C′ and D′ as points of the other system; but also C and D′ as the corresponding points of the one, and D and C′ as the corresponding points of the other system. Hence the lines CD and C′D′, as well as the lines CD′ and C′D, can be considered as corresponding lines, and both points of intersection Q and Q′ are consequently in the collinear axis AB. If now the line PDD′ revolve about P until it coincides with PCC′, the lines CD and C′D′ will then become tangents at C and C′, and these tangents must intersect in the collinear axis AB.

From this we derive a method of drawing tangents to a conic section from a given point; but as the method is well known we think it unnecessary to enter into further detail on this subject.

This may be shown analytically in the following manner.

Let $t_1\ u_1$ be the coordinates of a point C, and x_1y_1 those of a corresponding point C′ of the curve whose equation is

$$au^2+2btu+ct^2+2du+2et+f=0.$$

Then the equations of the tangents at the points C and C′ are

$$(au_1+bt_1+d)u+(bu_1+ct_1+e)t+du_1+et_1+f=0 \ldots\ldots (6)$$

$$(ay_1+bx_1+d)y+(by_1+cx_1+e)x+dy_1+ex_1+f=0 \ldots\ldots (7)$$

Now if these tangents are corresponding lines, the one equation must flow from the other by the substitution of the values of u and t as deduced in (3); hence recollecting that t_1u_1 and x_1y_1 are corresponding points, we have

$$u=\frac{-fy}{2dy+2ex+f},\quad t=\frac{-fx}{2dy+2ex+f};$$

$$u_1=\frac{-fy_1}{2dy_1+2ex_1+f},\quad t_1=\frac{-fx_1}{2dy_1+2ex_1+f};$$

and if these values of u, t, u_1, t_1 be written in equation (4), the resulting equation is identical with (5); therefore these tangents are corresponding lines, and consequently they will intersect in the collinear axis AB.

XIII.

Two given lines of the second degree can always be regarded as collinear figures. For let

$$au^2+2btu+ct^2+2du+2et+f=0 \ldots\ldots (1)$$

$$a'y^2+2b'xy+c'x^2+2d'y+2e'x+f'=0 \ldots\ldots (2)$$

be the equations of the two given curves; then if we write for u and t in (1) the values

$$u=\frac{m'y+n'x+p'}{my+nx+1},\quad t=\frac{m''y+n''x+p''}{my+nx+1},$$

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the resulting equation will be of the second degree in y and x, and which may therefore be made identical with (2). We shall thus have five equations of condition involving the eight quantities m', n', p', m'', n'', p'', m, and n; and therefore three of these quantities may have any arbitrary values, provided the values of the remaining five magnitudes do not involve imaginary expressions.

To illustrate this let us resolve the following problem.

Two given lines of the second degree are touched by two given straight lines inclined to each other at a given angle: find under what conditions both curves may be regarded as collinear and collinearly situated figures, and the intersection of the given lines as the collinear centre.

Let (1) and (2) be the equations of the two conic sections referred to the same coordinate axes. Now if in (1) we put for u and t the values

$$u=\frac{py}{my+nx+1},\quad t=\frac{px}{my+nx+1};$$

then the resulting equation is

$$(ap^2+2dmp+fm^2)y^2+2(bp^2+dnp+emp+fmn)xy+(cp^2+2enp+fn^2)x^2$$
$$+2(dp+fm)y+2(ep+fn)x+f=0.....(3)$$

which expresses a curve collinearly related to (1), and collinearly situated with it, the origin of coordinates of both being the common collinear centre. Now if this equation (3) is to be identical with (2) we get the following five equations of condition, viz.

$$(ap^2+2dmp+fm^2)f'=a'f.....(4)\qquad (dp+fm)f'=d'f.....(7)$$
$$(cp^2+2enp+fn^2)f'=c'f.....(5)\qquad (ep+fn)f'=e'f.....(8)$$
$$\{bp^2+(dn+em)p+fmn\}f'=b'f..(6)$$

From (7, 8) we obtain

$$m=\frac{d'}{f'}-p.\frac{d}{f};\ n=\frac{e'}{f'}-p.\frac{e}{f},$$

and these values of m and n reduce (6) to

$$p^2=\frac{f^2}{f'^2}.\frac{b'f'-d'e'}{bf-de}.$$

Let $h=bf-de$, and $h'=b'f'-d'e'$; then the last equation gives

$$p=\frac{f}{f'}.\left(\frac{h'}{h}\right)^{\frac{1}{2}}.....(9)$$

$$m=\frac{d'\sqrt{h}-d\sqrt{h'}}{f'\sqrt{h}}....(10);\quad n=\frac{e'\sqrt{h}-e\sqrt{h'}}{f'\sqrt{h}}.....(11)$$

and these values being substituted in (4, 5) give

$$(af-d^2)h'-(a'f'-d'^2)h=0.....(12)$$
$$(cf-e^2)h'-(c'f'-e'^2)h=0.....(13)$$

We must now ascertain whether equations (12, 13) can be satisfied if the point of intersection of the two tangential lines be taken as the origin of coordinates, and also whether the values of p, m, n as found in (9, 10, 11) be real or imaginary. Let then the point of intersection of the touching

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lines be the origin, and the lines themselves the axes of coordinates. The equations of these lines are therefore

$$u = 0, \quad t = 0; \quad \text{or } y = 0, \quad x = 0;$$

and if these lines touch the curves (1) and (2), the equations

$$au^2 + 2du + f = 0, \qquad a'y^2 + 2d'y + f' = 0,$$
$$ct^2 + 2et + f = 0, \qquad c'x^2 + 2e'x + f' = 0,$$

must have equal roots; consequently we have the conditions

$$\left.\begin{aligned} af - d^2 &= 0, & a'f' - d'^2 &= 0 \\ cf - e^2 &= 0, & c'f' - e'^2 &= 0 \end{aligned}\right\} \ldots\ldots\ldots (14);$$

and hence equations (12) and (13) are satisfied by the assumed coordinates. The values of p, m, n will be real if $\frac{h'}{h}$ is positive, and this depends on the position of the curves with reference to the axes of coordinates.

Now both curves may touch the coordinates axes on the positive side of the origin, or one of the curves may touch the axes on the positive side, and the other on the negative side; or again one curve may touch the axes on the positive side, and the other may touch one axis on the positive side, and the other on the negative. In other words, the coordinates of the points of contact of both curves may be positive, or the coordinates of the points of contact of one curve may be positive and those of the points of contact of the other negative, or whilst the coordinates of the points of contact of one curve are positive, the points of contact of the other may one be positive and the other negative. If therefore a straight line be drawn through the origin, it will intersect both curves or neither in the first two cases alluded to, and in the latter it will intersect the one and not the other. Let the equation of a line through the origin be

$$u = at, \quad \text{or } y = ax,$$

and eliminate u and y from equations (1) and (2); then we get

$$(aa^2 + 2ba + c)t^2 + 2(da + e)t + f = 0,$$
$$(a'a^2 + 2b'a + c')x^2 + 2(d'a + e')x + f' = 0.$$

The roots of these equations are real or imaginary according as the expressions

$$(da + e)^2 - f(aa^2 + 2ba + c), \quad (d'a + e')^2 - f'(a'a^2 + 2b'a + c')$$

are positive or negative. But by (14) these expressions are reduced to

$$2(de - bf)a; \qquad 2(d'e' - b'f')a;$$

or to, $\qquad -2ha; \qquad -2h'a.$

Hence if h and h' have like signs, both curves will be intersected or not intersected by the straight line drawn through the origin, and if they have different signs, the line will only cut one of the curves. In other words $\frac{h'}{h}$ is positive in the first two positions of the curves, and negative in the latter position. Consequently the values of p, m, n are real in the former cases and imaginary in the latter; and therefore it is obvious that two curves of the second degree which touch two given straight lines can always be regarded as collinear and collinearly situated figures, if both curves are on the same side of the origin, or one of the curves on one side of the origin,

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and the other on the other or vertically opposite side. The point of intersection of the touching lines is the collinear centre, and it is so in a double point of view, because the quantities $\sqrt{h}$ and $\sqrt{h'}$ may be either positive or negative, and hence two values can be assigned to each of the quantities p, m, n.

Substituting the double values of the quantities p, m, n, in the equation of the collinear axis, viz.

$$my + nx + 1 - p = 0,$$

we get either

$$(d'\sqrt{h} - d\sqrt{h'})y + (e'\sqrt{h} - e\sqrt{h'})x + f'\sqrt{h} - f\sqrt{h'} = 0 \ldots\ldots (15)$$

or $(d'\sqrt{h} + d\sqrt{h'})y + (e'\sqrt{h} + e\sqrt{h'})x + f'\sqrt{h} + f\sqrt{h'} = 0 \ldots\ldots (16)$

which are the equations of the collinear axes. Adding and subtracting these equations, we get

$$d'y + e'x + f' = 0, \qquad dy + ex + f = 0,$$

which are the polars of the origin or collinear centre, and consequently the chords of contact of the axes of coordinates with both curves.

Hence it follows that both collinear axes intersect in these chords of contact, and therefore the point of intersection of the collinear axes coincides with the point of intersection of the chords of contact.

If one of the curves cuts one or both collinear axes, the other curve also cuts one or both collinear axes in the same points.

For to a point in the collinear axis of one system, there is a corresponding point in the collinear axis of the other system, and if both systems are collinearly situated, these points are coincident. This may be shown also in the following way.

From equations (14) we get

$$a = \frac{d^2}{f}, \; a' = \frac{d'^2}{f'}, \; c = \frac{e^2}{f}, \; c' = \frac{e'^2}{f'},$$

and these values of a, a', c, c' being written in (1) and (2) reduce these equations (t and u being replaced by x and y) to the forms

$$(dy + ex + f)^2 + 2hxy = 0 \ldots\ldots\ldots\ldots\ldots (17)$$

$$(d'y + e'x + f')^2 + 2h'xy = 0 \ldots\ldots\ldots\ldots\ldots (18)$$

Now the values of the coordinates of the point of intersection of these curves which satisfy both (17) and (18) when the former is multiplied by h' and the latter by h, will also satisfy the difference of the same equations when respectively multiplied by h and h'; and hence they will satisfy the equation

$$h(d'y + e'x + f')^2 - h'(dy + ex + f)^2 = 0.$$

But this equation is the same as

$$\{\sqrt{h}(d'y + e'x + f') - \sqrt{h'}(dy + ex + f)\}$$
$$\times \{\sqrt{h}(d'y + e'x + f') + \sqrt{h'}(dy + ex + f)\} = 0 \ldots\ldots (19)$$

which is evidently the product of the collinear axes (15, 16); hence the collinear axes pass through the intersections of both curves.

In a similar manner it may be shown that two confocal curves of the second degree can be regarded as collinear and collinearly situated figures, and the common focus as the collinear centre.

(To be continued.)

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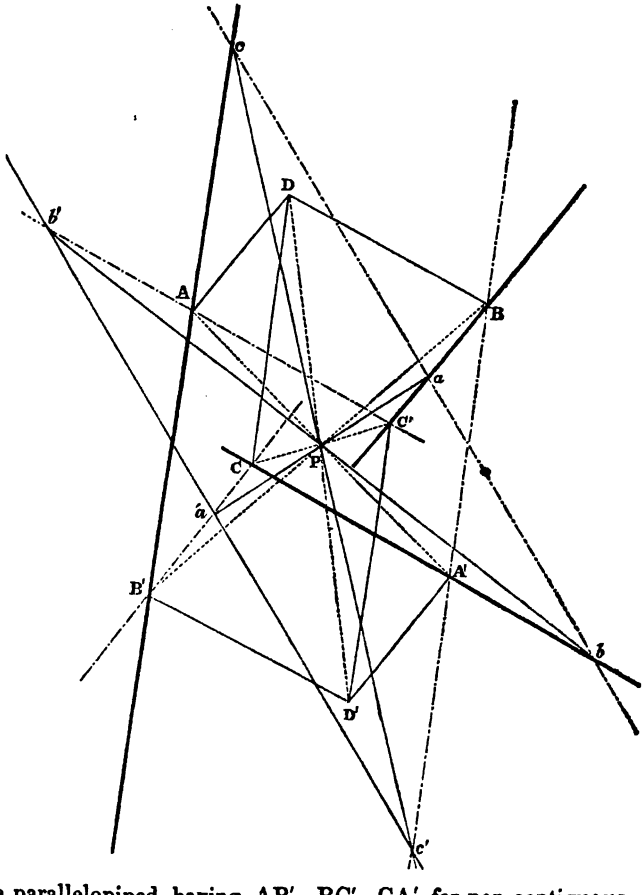
MODERN GEOMETRY.

[*T. S. Davies, F. R. S.*]

(*Continued from p. 95.*)

XXXII.

Upon any three lines in space, no two of which are in the same plane, nor all three parallel to the same plane, a parallelopiped may be constituted having these lines as three non-contiguous and non-parallel edges; and there can be only one such parallelopiped constituted on the same three lines.



Let AB′, BC′, CA′ be the three given lines, unlimited in their extensibility, no two of which are in the same plane nor all three parallel to the same plane.

Through AB′ one plane, and one only, can be drawn parallel to BC′; and through BC′ one plane, and one only, parallel to AB′. These planes, also, as is well known, are parallel to each other. In like manner, through BC′ and CA′ a single pair of parallel planes may be drawn; and through CA′, AB′ another single pair.

The three pairs of parallel planes thus constituted will by their mutual intersections form a parallelopiped, having AB′, BC′, CA′ for non-contiguous and non-parallel edges.

One such parallelopiped can therefore be always constituted on the three lines, as stated in the proposition. Also, since the pairs of parallel planes through the lines AB′, BC′, CA′ are unique, the parallelopiped formed by them is, of necessity, unique.

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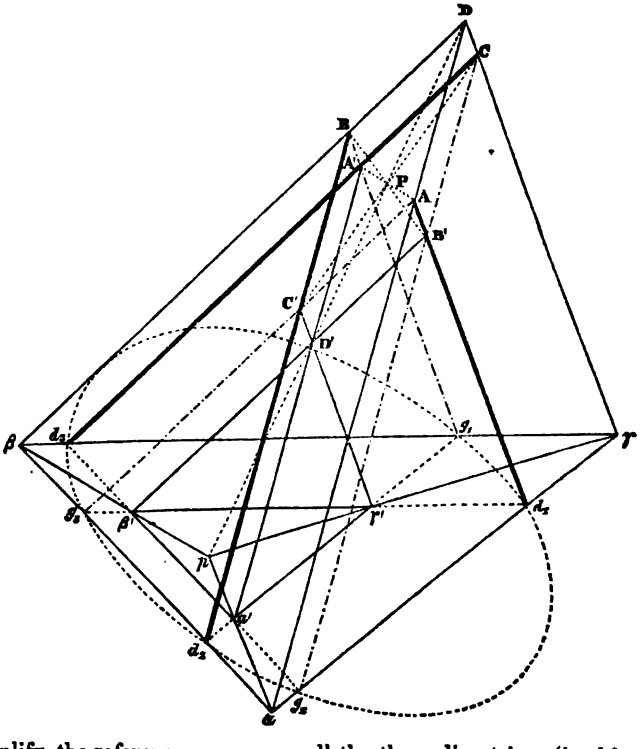
Draw the diagonals AA', BB', CC', DD' of the parallelopiped: these, as is known, all meet in a point, P, and bisect each other there. Then, of the parallel edges to the three given lines, those which join the extremities of the same diagonals (DD' being, for the present excluded) have this property:—*of each resting upon two of the directrices of the curve surface generated by a line moving along them, and being parallel to the third directrix.* Thus, B'C rests upon AB' and CA' and is parallel to BC'; C'A rests upon AB' and BC' and is parallel to CA'; A'B rests upon BC' and CA' and is parallel to AB'. These are the three lines referred to in the foot note to art. XXVIII: and they have received the name of *symmetrical edges*; so that AB', BA' are symmetrical to one another, as are likewise BC' and C'A symmetrical respectively to B'C and CA'.

The six edges in question form a twisted hexagon, *(hexagon gauche)* whose opposite sides are respectively equal and parallel. The alternate sides are successively a directrix and a generatrix; and these are interchangeable in virtue of art. XXX.

XXXIII.

If the six edges of the twisted hexagon, formed as in the preceding paragraph, be cut by any plane, the six points of intersection will lie in a conic section.

The parallelopiped being completed, as in the last figure, where AB'CA'BC' is the hexagon, let the planes through D and D' be drawn to cut the given plane. Let $\alpha\beta\gamma$ be the triangle formed by the traces of those about D, and $\alpha'\beta'\gamma'$ that formed similarly by those about D'. Then since each of the planes about D' is parallel to a corresponding one of those about D, the sides of the two triangles will be parallel, each to each.



Now, to simplify the references we may call the three directrices (in this fig. and that to preceding article) AB', BC', CA' by the names d_1, d_2, d_3;

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and the generatrices A'B, B'C, C'A by the names g_1, g_2, g_3: so that the directrices d_1, d_2 are touched by g_3; d_2, d_3 by g_1; and d_3, d_1 by g_2.

This being done, we see that g_1 is the intersection of the planes ADC and A'D'B', and hence its trace on the cutting plane is the intersection of the traces of these planes; or g_1 the trace of A'B on the plane in the figure above is the intersection of the traces $a\cdot\gamma$, $a'\beta'$ of the planes themselves. In the same way it may be shewn that the traces of all the sides of the twisted hexagon on the cutting plane are the intersections of the sides of the two triangles $a\beta\gamma$, $a'\beta'\gamma'$, as expressed in the figure. It may, however, facilitate the reader's obtaining a clear view of this fact, if we put down in a tabular form the derivation of the whole system.

Generatrix	Directrix	is the intersection of		Generatrix	Directrix	is the intersection of
g_1		BDC, C'D'A'		g_1		$\beta\gamma$, $\gamma'a'$
g_2		CDA, A'D'B'		g_2		γa, $a'\beta'$
g_3		ADB, B'D'C'	Whence, the trace of	g_3		$a\beta$, $\beta'\gamma'$
	d_1	CDA, B'D'C'			d_1	γa, $\beta'\gamma'$
	d_2	ADB, C'D'A'			d_2	$a\beta$, $\gamma'a'$
	d_3	BDC, A'D'B'			d_3	$\beta\gamma$, $a'\beta'$

Now viewing the triangle $a\beta\gamma$ as cut by the transversals $a'\beta'$, $\beta'\gamma'$, $\gamma'a'$ parallel to its sides, we have

$$\gamma g_2 . d_2\beta = g_2 a . \gamma d_2 \text{ from transv. } a'\beta',$$
$$a g_3 . d_1\gamma = g_3\beta . a d_1 \ldots\ldots\ldots \beta'\gamma',$$
$$a d_2 . g_1\beta = d_2\beta . \gamma g_1 \ldots\ldots\ldots \gamma'a'.$$

Wherefore, compounding these we get,

$$a d_2 . a g_3 . \beta d_3 . \beta g_1 . \gamma d_1 . \gamma g_2 . = a g_2 . a d_1 . \gamma d_3 . \gamma g_1 . \beta d_2 . \beta g_3 .$$

Which, (*Converse of Hutton*, ii. pr. 26, p. 237) shews that the six points d_1, d_2, d_3, g_1, g_2, g_3 are in a conic section.*

Scholium. The triangles $a\beta\gamma$, $a'\beta'\gamma'$ are copolar, and their common pole is p the intersection of DD' with the cutting plane; as is too evident to need a formal demonstration.

XXXIV.

Let any generatrix abc be taken with respect to the directrices AB', BC', CA', ***resting upon them at the points a, b, c: then another a'b'c' may be made to rest upon their symmetrical directrices*** B'C, C'A, A'B,

* The property now deduced is only a particular case of a more general theorem:—*if two triangles be copolar, their mutual intersections will be six points in the same conic section.*

For two copolar triangles are coaxial (*Hutton*, ii. p. 221.) Whence by their mutual intersections they give six points situated in the same conic section, by the converse of Pascal's theorem.

This proof does not, however, apply in the extreme case in the text, since the axis is thrown infinitely distant. Whence the necessity for the modified demonstration given above.

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in a', b',c' : such that the lines abc, $a'b'c'$ shall be parallel, equidistant from P, *and symmetrically situated with respect to that point.*

(See figure in page 199.)

Draw aP : then since B'C, BC' are parallel they are in one plane; and P being in that plane, aP is so likewise, and will hence cut B'C in a point a'. In the same manner if bP, cP be drawn, they will meet C'A, A'B in b' and c'. Also, since aP, bP, cP are in the plane drawn through the point P and line abc, the seven points a, b, c, a', b', c' and P are in the same plane.

Also, since B'C, BC' are equal and parallel, the triangles CPB', C'PB are in all respects equal; and hence, obviously, $Ca' = C'a$, $C'a = Ca'$, and $a'P = Pa$. And similarly, in the corresponding pairs of triangles we have $b'P = Pb$, and $c'P = Pc$. Whence, it readily follows that a', b', c' are in one straight line parallel and symmetrical to abc.

XXXV.

Since A'B rests upon the two directrices A'C, BC' and is parallel to the third AB' it is one of the generatrices meeting the third directrix at an infinite distance. It is therefore situated upon the hyperboloïd. In the same manner, B'C and C'A are also generatrices of the hyperboloïd : and they form an unique system.

Now any three of the generatrices may be taken as directrices, as has been shewn in (XXX), and hence if we make the interchange in the case before us, AB', B'C, C'A will become the three corresponding generatrices, forming also the corresponding unique system.

It, therefore, follows that for every generatrix of the first mode, abc, there is one of the second mode, $a'b'c'$, parallel to it and symmetrically situated. From this we see that the surface is altogether symmetrical with respect to the parallelopiped formed upon its three defining directrices as non-contiguous and non-parallel edges; and that the surface has a centre, P, with respect to which it is perfectly symmetrical.

Again, since for every line abc of the first genesis there is another $a'b'c'$ parallel to it in the second, the plane diametral sections along any one of the generatrices, drawn through the centre, is composed of two indefinitely extended parallel lines, equidistant from the centre.

Moreover, if any three new directrices be taken, which are in positions *consistent with the original ones* (that is, which would have given the surfaces, or which is the same thing, would be generatrices of the second system founded on the first), the point P will remain unaltered, though the angular points and the edges of the resulting parallelopiped will be altered by this change. For the six edges will be symmetrical and parallel in the same manner that the two corresponding ones abc, $a'b'c'$ were proved to be.

XXXVI.

Instead of the parallelopiped on DD' let us conceive that another is constituted on dd', these points being on DD' produced or not, and equidistant from P, and the planes through d, d' parallel to those through D, D' respectively : then this new parallelopiped will be similar and similarly situated to the original one, and the six edges which constitute the new twisted hexagon will be parallel to the corresponding ones of the original hexagon, and similarly situated with respect to the diagonal.

The triangles in which the planes through d and d' meet the cutting plane will have their sides parallel to those of $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ respectively, and will have p for their pole. They will also approach or recede from $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ by

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equal quantities estimated on ap, βp, γp respectively; and hence the entire figure composed from these triangles will be similar and similarly situated with respect to ap, βp, γp, whatever be the length of dd'. Whence the conic sections which pass through the six intersections of the edges of the variable twisted hexagon are always similar and similarly situated with respect to $p\alpha$, $p\beta$, $p\gamma$.

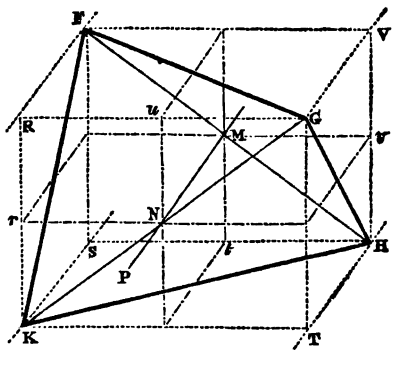
Again, from very similar reasoning it follows that if the original twisted hexagon be cut by any plane parallel to that which contains $\alpha\beta\gamma$, the conic section through the intersections of that plane with the six edges will always be similar and similarly situated with respect to the pole and its corresponding radiants.

XXXVII.

The preceding properties are intended to shew that there are several aspects under which the surface, distorted as it at first appears, becomes possessed of regular features. Before proceeding to shew that *all* the generatrices intersect the plane in the conic section already determined, it may be worth while to add one other important property indicative of the regularity of the surface.

Take any one of the twisted quadrilaterals into which the surface is divided by the directrices and generatrices: the line which bisects its diagonals always passes through the centre, P, *of the parallelopiped upon* DD'.

Let FGHK be the quadrilateral, whose diagonals are bisected in M and N respectively: it is to be shewn that MN will pass through the centre, P, of the parallelopiped described on three directrices of the hyperboloïd of one sheet, of which either FG, KH or KF, GH are two.



Through FG, KH draw the parallel planes RV, ST; and through KF, GH the parallel planes RS, VT. These pairs of planes are, as is well known, unique.

Now the lines FH, GK which form the diagonals, (lying between the parallel planes RS, VT, and being likewise bisected in M and N), the points M, N are situated in a plane tu equidistant from RS, VT and parallel to them.

In the same way it is shewn that the points M, N are situated in a plane rv equidistant from RV, ST and parallel to them.

Whence, being in two planes which are respectively parallel to the two pairs of planes, they are in the intersection of ut, rv.

But RV, ST are two faces of the parallelopiped depending on FG, KH as directrices in one mode of genesis, and hence the plane rv passes through the centre P of that parallelopiped. In the same manner KF, GH being taken as two of the directrices of the second system, the plane tu passes through the centre of the corresponding parallelopiped, which is the same with the former one P. Both planes, therefore, pass through P; and hence their intersection MN also passes through P, as affirmed in the enunciation.

(To be continued.)

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ON A NEW METHOD OF INTEGRATING LINEAR DIFFERENTIAL EQUATIONS.

[*The Rev. Brice Bronwin.*]

The method of integrating certain linear differential equations exemplified in this paper is of a novel character and easy of application. Before entering upon it, I must premise four subsidiary formulæ, which will be in constant requisition; observing that throughout this paper D stands for $\frac{d}{dx}$.

$$\left.\begin{aligned} xD^n y &= D^n xy - nD^{n-1} y \\ x^2 D^n y &= D^n x^2 y - 2nD^{n-1} xy + n(n-1)D^{n-2} y \end{aligned}\right\} \ldots\ldots\ldots (A)$$

$$\left.\begin{aligned} xD^{-n} y &= D^{-n} xy + nD^{-n-1} y \\ x^2 D^{-n} y &= D^{-n} x^2 y + 2nD^{-n-1} xy + n(n+1)D^{-n-2} y \end{aligned}\right\} \ldots\ldots\ldots (B)$$

These formulæ are easily verified by differentiation; or by putting $D^{-n+1}y$, $D^{-n+2}y$, $D^{n+1}y$, $D^{n+2}y$ respectively for y and differentiation. They are derived successively from the case of $n = 1$ by putting Dy or $D^{-1}y$ for y, and reducing by what goes before. We might add to them similar values of $x^3 D^n y$, $x^3 D^{-n} y$, *etc.* It is worthy of remark that changing (n) into $(-n)$ converts (A) into (B). We will now proceed to some examples.

Let $$(1 - x^2)\frac{d^2 y}{dx^2} + p(p+1)y = 0 \ldots\ldots\ldots\ldots\ldots (1)$$

or, $D^2 y - x^2 D^2 y + p(p+1)y = 0$. Make $y = D^{p-1} u$, and the last equation becomes $D^{p+1} u - x^2 D^{p+1} u + p(p+1)D^{p-1} u = 0$. Or putting for $x^2 D^{p+1} u$, its value from (A), and omitting the terms which destroy one another,

$$D^{p+1} u - D^{p+1} x^2 u + (2p+2)D^p xu = 0;$$

or $$Du - Dx^2 u + (2p+2)xu = D^{-p} 0 = CD^{-p+1}.$$

This will give the complete integral at once; but it may be expressed more simply if the two particular integrals are found separately. We shall therefore not operate upon (0), but take

$$Du - Dx^2 u + (2p+2)xu = 0; \text{ or } (1 - x^2)\frac{du}{dx} + 2pxu = 0,$$

which gives $$u = C(1 - x^2)^p, \quad y = C\left(\frac{d}{dx}\right)^{p-1}(1 - x^2)^p.$$

Again, make $y = D^{-p-2} u$; there results $D^{-p} u - x^2 D^{-p} u + p(p+1)D^{-p-2} u = 0$.

Put for $x^2 D^{-p} u$ its value from (B), and we have

$$D^{-p} u - D^{-p} x^2 u - 2pD^{-p-1} xu = 0;$$

or multiplying by D^{p+1}, $$Du - Dx^2 u - 2pxu = 0,$$

which is equivalent to $$(1 - x^2)\frac{du}{dx} - (2p+2)xu = 0.$$

This gives $u = C_1(1 - x^2)^{-p-1}$, $y = C_1\left(\frac{d}{dx}\right)^{-p-2}(1 - x^2)^{-p-1}$.

Therefore the complete integral of (1) is

$$y = C\left(\frac{d}{dx}\right)^{p-1}(1 - x^2)^p + C_1\left(\frac{d}{dx}\right)^{-p-2}(1 - x^2)^{-p-1}.$$

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Next let $x^2\frac{d^2y}{dx^2}+m\frac{dy}{dx}-p(p+1)y=0$ .....(2)

Or $x^2D^2y+mDy-p(p+1)y=0$. Here also make $y=D^{p-1}u$; then $x^2D^{p+1}u+mD^pu-p(p+1)D^{p-1}u=0$. Put in this for $x^2D^{p+1}u$, its value from (A), and there results

$$D^{p+1}x^2u-(2p+2)D^pxu+mD^pu=0,$$

or, $$Dx^2u-(2p+2)xu+mu=0,$$

or, $$x^2\frac{du}{dx}+(m-2px)u=0.$$

Therefore $u=Cx^{2p}\epsilon^{\frac{m}{x}}$, $y=C\left(\frac{d}{dx}\right)^{p-1}x^{2p}\epsilon^{\frac{m}{x}}$.

Again, make $y=D^{-p-2}u$, and we have

$$x^2D^{-p}u+mD^{-p-1}u-p(p+1)D^{-p-2}u=0.$$

Put for $x^2D^{-p}u$, its value from (B), and we find

$$D^{-p}x^2u+2pD^{-p-1}xu+mD^{-p-1}u=0,$$

or $Dx^2u+(m+2px)u=0$; and $x^2\frac{du}{dx}+\{m+(2p+2)x\}u=0$.

Hence $u=C_1x^{-2p-2}\epsilon^{\frac{m}{x}}$, $y=C_1\left(\frac{d}{dx}\right)^{-p-2}x^{-2p-2}\epsilon^{\frac{m}{x}}$; and the complete integral of (2) is $y=C\left(\frac{d}{dx}\right)^{p-1}x^{2p}\epsilon^{\frac{m}{x}}+C_1\left(\frac{d}{dx}\right)^{-p-2}x^{-2p-2}\epsilon^{\frac{m}{x}}$.

It perhaps is scarcely necessary to observe, that in this example we might have found the complete integral at once; but it would have been complex. I shall only add one more example.

Let $x^2\frac{d^2y}{dx^2}+(m-x)x\frac{dy}{dx}+n(m-n-1)y=0$ (n an integer)....(3)

Or $$x^2D^2y+mxDy-x^2Dy+n(m-n-1)y=0.$$

Make $y=x^{-n}D^{-n-1}u$, then $Dy=x^{-n}D^{-n}u-nx^{-n-1}D^{-n-1}u$,

$D^2y=x^{-n}D^{-n+1}u-2nx^{-n-1}D^{-n}u+n(n+1)x^{-n-2}D^{-n-1}u$.

Substituting these values and reducing, we find

$$xD^{-n+1}u-xD^{-n}u+(m-2n)D^{-n}u+nD^{-n-1}u=0;$$

which by (B) becomes

$$D^{-n+1}xu-D^{-n}xu+(m-n-1)D^{-n}u=0;$$

or, $Dxu+(m-n-1-x)u=0$; and $x\frac{du}{dx}+(m-n-x)u=0$,

which gives $u=Cx^{n-m}\epsilon^x$, $y=Cx^{-n}\left(\frac{d}{dx}\right)^{-n-1}x^{n-m}\epsilon^x$.

Before we can find the most simple expression for the other particular integral, we must transform (3).

In (3) make $y=x^{-m}\epsilon^xz$, and it will transform into

$$x^2\frac{d^2z}{dx^2}+(x-m)x\frac{dz}{dx}+(m-n)(n+1)z=0.....(4)$$

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Or, $x^2D^2z + x^2Dz - mxDz + (m-n)(n+1)z = 0.$

Make $z = x^{n+1}D^nu$; then $Dz = x^{n+1}D^{n+1}u + (n+1)x^nD^nu$;

$$D^2z = x^{n+1}D^{n+2}u + (2n+2)x^nD^{n+1}u + n(n+1)x^{n-1}D^nu.$$

With these values of z, Dz, and D^2z, the above will become

$$xD^{n+2}u + xD^{n+1}u + (2n+2-m)D^{n+1}u + (n+1)D^nu = 0;$$

which by (A) will reduce to

$$D^{n+2}xu + D^{n+1}xu + (n-m)D^{n+1}u = 0;$$

or $$Dxu + (n-m+x)u = 0;$$

or $$x\frac{du}{dx} - (m-n-1-x)u = 0, \text{ and } u = C_1x^{m-n-1}\epsilon^{-x},$$

$$z = C_1x^{n+1}\left(\frac{d}{dx}\right)^n x^{m-n-1}\epsilon^{-x}, \quad y = C_1x^{n-m+1}\epsilon^x\left(\frac{d}{dx}\right)^n x^{m-n-1}\epsilon^{-x}.$$

Therefore the complete integral of (3) is

$$y = Cx^{-n}\left(\frac{d}{dx}\right)^{-n-1} x^{n-m}\epsilon^x + C_1x^{n-m+1}\epsilon^x\left(\frac{d}{dx}\right)^n x^{m-n-1}\epsilon^{-x}.$$

And $z = x^m\epsilon^{-x}y$ will give the complete integral of (4).

In integrating differential equations, we have frequently to divide by an operating factor, and to find its effect on the second member of the equation; for which purpose we sometimes want a more general formula than writers give, which I shall here investigate.

Suppose ϕ and λ any functions of x, we easily find that

$$\epsilon^{-\int\frac{\lambda}{\phi}dx}(\phi D)\epsilon^{\int\frac{\lambda}{\phi}dx}y = (\phi D + \lambda)y.$$

This is verified by actually performing the operation (ϕD) in the first member. For y put $\epsilon^{-\int\frac{\lambda}{\phi}dx}(\phi D)\epsilon^{\int\frac{\lambda}{\phi}dx}y$ in the first member, and its equal $(\phi D + \lambda)y$ in the second; there results

$$\epsilon^{-\int\frac{\lambda}{\phi}dx}(\phi D)^2\epsilon^{\int\frac{\lambda}{\phi}dx}y = (\phi D + \lambda)^2y.$$

In like manner $\epsilon^{-\int\frac{\lambda}{\phi}dx}(\phi D)^3\epsilon^{\int\frac{\lambda}{\phi}dx}y = (\phi D + \lambda)^3y$, *etc.*

Also from above, $y = (\phi D + \lambda)^{-1}\epsilon^{-\int\frac{\lambda}{\phi}dx}(\phi D)\epsilon^{\int\frac{\lambda}{\phi}dx}y.$

Put in this $\epsilon^{-\int\frac{\lambda}{\phi}dx}(\phi D)^{-1}\epsilon^{\int\frac{\lambda}{\phi}dx}y$ for y, and it becomes

$$\epsilon^{-\int\frac{\lambda}{\phi}dx}(\phi D)^{-1}\epsilon^{\int\frac{\lambda}{\phi}dx}y = (\phi D + \lambda)^{-1}y.$$

And so on for higher negative powers. Multiplying these results, continued to infinity, by any constants, and taking the sum; we have

$$\epsilon^{-\int\frac{\lambda}{\phi}dx}f(\phi D)\epsilon^{\int\frac{\lambda}{\phi}dx}y = f(\phi D + \lambda)y \ldots\ldots (C)$$

Suppose $(\phi D + \beta)(\Psi D + \lambda)y = 0$. We may divide first by $\phi D + \beta$, and operate with it upon (0) in the second member; and then we may

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proceed with $\Psi D + \lambda$ in like manner. The formula (C) enables us to do this. It is to be observed that ϕ, β, Ψ, and λ are all supposed to be functions of x. By the above process we have

$$(\phi D + \beta)(\Psi D + \lambda)y = \epsilon^{-\int \frac{\beta}{\phi} dx} (\phi D) \epsilon^{\int \left(\frac{\beta}{\phi} - \frac{\lambda}{\Psi}\right) dx} (\Psi D) \epsilon^{\int \frac{\lambda}{\Psi} dx} y.$$

The second member of this made equal to (0) is easily integrated.

We may also make $(\Psi D + \lambda)y = u$, and then $(\phi D + \beta)u = 0$. Integrating the latter of these first, and then the former, we find y. By any of these methods we may integrate the preceding equation, and also any equation of a higher order when of a similar form. It must be observed that we cannot change the order of the factors $\phi D + \beta$, $\Psi D + \lambda$ without changing the equation.

But if we have an equation of the form $f(\phi D + \lambda)y = 0$, and we make r_1, r_2, *etc.* the roots of $f(x) = 0$, we may put it under the form

$$(\phi D + \lambda - r_1)(\phi D + \lambda - r_2) \ldots . y = 0.$$

Here, as the product of the factors is the same, whatever be their order, the equation will remain the same when the order is changed. If therefore we divide by all of them but one, without operating upon the second member, we shall have the equations

$$(\phi D + \lambda - r_1)y = 0, \quad (\phi D + \lambda - r_2)y = 0, \text{ etc};$$

which will each give a particular integral. This is Mr. BOOLE's method. But here we may make the first member of (C) equal to (0), and perhaps integrate it more easily.

We now proceed to another formula connected with this subject. If D operate upon y only, we have $\epsilon^{\phi(D)} x \epsilon^{-\phi(D)} y = xy$. Let D′ operate upon x only, and multiply both sides by $\epsilon^{\phi(D+D')-\phi(D)}$; then

$$\epsilon^{\phi(D+D')} x \epsilon^{-\phi(D)} y = \epsilon^{\phi(D+D')-\phi(D)} xy = \epsilon^{\phi'(D)D' + \frac{1}{2}\phi''(D)D'^2 + \ldots} xy$$
$$= \{1 + \phi'(D)D' + \tfrac{1}{2}\phi''(D)D'^2 + \tfrac{1}{2}\phi'(D)^2 D'^2 + etc.\} xy$$
$$= xy + \phi'(D)y = (x + \phi'(D))y.$$

We may change D+D′ into D in the first member, D operating upon both x and y or upon all that follows it. Thus $\epsilon^{\phi(D)} x \epsilon^{-\phi(D)} y = \{x + \phi'(D)\}y$. Change y into $\epsilon^{\phi(D)} x \epsilon^{-\phi(D)} y$, in the first member, and into its equal $\{x + \phi'(D)\}y$ in the second; and we have $\epsilon^{\phi(D)} x^2 \epsilon^{-\phi(D)} y = \{x + \phi'(D)\}^2 y$. And proceeding exactly as in the investigation of (C), we find

$$\epsilon^{\phi(D)} f(x) \epsilon^{-\phi(D)} y = f\{x + \phi'(D)\}y \ldots\ldots\ldots (D)$$

This is due to Mr. BOOLE; but he has arrived at it in a very different manner. If we put $a + x$ in the place of x, we have $\{1 + \phi'(D)D'\}(a + x)y = \{a + x + \phi'(D)\}y$. Therefore

$$\epsilon^{\phi(D)} f(a + x) \epsilon^{-\phi(D)} y = f\{a + x + \phi'(D)\}y \ldots\ldots (E)$$

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Again, if we put $a+bx+x^2$ for x, we have

$$\{1+\phi'(D)D'+\tfrac{1}{2}\phi''(D)D'^2+\tfrac{1}{3}\phi'(D)^2D'^2\}(a+bx+x^2)y=$$
$$\{a+bx+x^2+(b+2x)\phi'(D)+\phi''(D)+\phi'(D)^2\}y;$$ and therefore

$$\epsilon^{\phi(D)}f(a+bx+x^2)\epsilon^{-\phi(D)}y=$$
$$f\{a+bx+x^2+(b+2x)\phi'(D)+\phi''(D)+\phi'(D)^2\}y.....(F)$$

And thus we might proceed to more complicated functions. The formulæ (D), (E), and (F) may be employed for the purposes of integration by integrating that member which is easiest to integrate, which will depend upon the form of $\phi(D)$. The other member will serve to find the equation.

In $\{x+\phi'(D)\}y=\epsilon^{\phi(D)}x\epsilon^{-\phi(D)}y$, change $\phi(D)$ into $\log\phi(D)$, and y into $\phi(D)y$; then $\{x\phi(D)+\phi'(D)\}y=\phi(D)xy$. Change y in the first member of this into $\{x\phi(D)+\phi'(D)\}y$, and in the second into $\phi(D)xy$, which gives $\{x\phi(D)+\phi'(D)\}^2y=\{\phi(D)x\}^2y$, and so on. Therefore as before

$$f\{x\phi(D)+\phi'(D)\}y=f\{\phi(D)x\}y.....(G)$$

In like manner we find

$$f\{(a+x)\phi(D)+\phi'(D)\}y=f\{\phi(D)(a+x)\}y.....(H)$$

To give an example of integration from (G), make $f(x)=2+x$, that is, let this be the form of the function f; and make $\phi(D)=D^2-D+1$. Then

$$\{xD^2+(2-x)D+1+x\}y=(D^2-D+1)xy+2y, \text{ and}$$
$$\{xD^2+(2-x)D-1+x\}y=(D^2-D+1)xy.$$

Make $(D^2-D+1)xy=0$, and let m_1, m_2 be the roots of $m^2-m+1=0$; we have $y=Cx^{-1}\epsilon^{m_1x}+C_1x^{-1}\epsilon^{m_2x}$ for the integral of $x\frac{d^2y}{dx^2}+(2-x)\frac{dy}{dx}$ $+(x-1)y=0$.

If $\phi(D)$ be an integer function, $\phi(D)f(x)y=0$ is always and easily integrable. Let D_1 operate upon x only, and D_2 upon y only; then $\phi(D)=\phi(D_1+D_2)=\phi(D_1)+\phi'(D_1)D_2+$ *etc.*, a series which will always terminate, and we shall have

$$\phi(D_1)f(x)y+\phi'(D_1)f(x)\frac{dy}{dx}+\tfrac{1}{2}\phi''(D_1)f(x)\frac{d^2y}{dx^2}+etc.=0,$$

which will be integrated by integrating $\phi(D)f(x)y=0$.

If $\phi_1(D)f_1(x)\phi(D)f(x)y=0$, both the integral and the expansion of this may be derived from those of the preceding by changing ϕ and f into ϕ_1 and f_1, and changing y into $\phi(D)f(x)y$, and expanding as before.

The first member of (D) may be written $\epsilon^{\phi(D_2+D_1)-\phi(D_2)}f(x)y=$ $\epsilon^{\phi'(D_2)D_1+\frac{1}{2}\phi''(D_2)D_1^2+etc.}f(x)y$, which expanded by the powers of D_1 will give a series that will terminate, $f(x)$ being an integer function of x. And if it be made equal to nothing, the integral of $f\{x+\phi'(D)\}y=0$ will be its integral.

An example or two of the application of this theory to the integration of partial differential equations might be given; but as I am afraid of my paper being too long, I shall here terminate it.

Gunthwaite Hall, March 17th, 1846.

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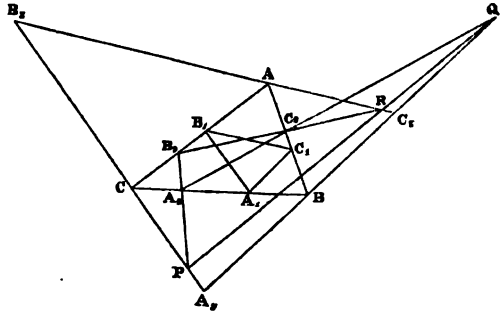
SOLUTIONS OF MATHEMATICAL EXERCISES.

LXXIV. *Mr. Thomas Weddle, Newcastle-upon-Tyne.*

Let lines be drawn from the angles of a triangle through two points and terminating in the opposite sides, then if the extremities of each set of lines be joined, two inscribed co-polar triangles will be formed. Also from the angular points of the primitive triangle let lines be drawn parallel to the corresponding sides of these triangles, and intersecting those of the other: it is required to prove that the three points of intersection range in the same straight line.

[FIRST SOLUTION. *Mr. Samuel Bills, Hawton; and similarly by the Proposer.*]

Let ABC be the primitive triangle, and $A_1B_1C_1$, $A_2B_2C_2$ the two inscribed copolar triangles. Draw AR, BQ, CP respectively parallel to B_1C_1, C_1A_1, A_1B_1 and let AR, BQ, CP intersect B_2C_2, C_2A_2, A_2B_2 in R,Q,P; then will the points P,Q,R be in the same straight line.



The triangles CPA_2 and CPB_2 give

$$\frac{PA_2}{PC} = \frac{\sin PCA_2}{\sin PA_2C} = \frac{\sin CA_1B_1}{\sin CA_2B_2},$$

$$\frac{PB_2}{PC} = \frac{\sin PCB_2}{\sin PB_2C} = \frac{\sin CB_1A_1}{\sin A_2B_2C},$$

$$\therefore \frac{PB_2}{PA_2} = \frac{\sin CB_1A_1}{\sin CA_1B_1} \cdot \frac{\sin CA_2B_2}{\sin A_2B_2C} = \frac{CA_1}{CB_1} \cdot \frac{CB^2}{CA_2}.$$

$$\text{Similarly, } \frac{QA_2}{QC_2} = \frac{BC_1}{BA_1} \cdot \frac{BA_2}{BC_2}, \text{ and } \frac{RC_2}{RB_2} = \frac{AB_1}{AC_1} \cdot \frac{AC_2}{AB_2};$$

and taking the product of these results, we have

$$\frac{PB_2 \cdot QA_2 \cdot RC_2}{PA_2 \cdot QC_2 \cdot RB_2} = \frac{CA_1 \cdot BC_1 \cdot AB_1}{CB_1 \cdot BA_1 \cdot AC_1} \cdot \frac{CB_2 \cdot BA_2 \cdot AC_2}{CA_2 \cdot BC_2 \cdot AB_2}.$$

But by the properties of copolar triangles, we have $CA_1 \cdot BC_1 \cdot AB_1 = CB_1 \cdot BA_1 \cdot AC_1$, and $CB_2 \cdot BA_2 \cdot AC_2 = CA_2 \cdot BC_2 \cdot AB_2$; hence we get

$$PB_2 \cdot QA_2 \cdot RC_2 = PA_2 \cdot QC_2 \cdot RB_2,$$

and this relation, since P,Q,R are on the sides of the triangle $A_2B_2C_2$, shows that the points P,Q,R are on the same straight line.

[SECOND SOLUTION. *Mr. G. W. Hearn; and similarly by Mr. R. H. Wright, London.*]

Let S and S_1 be the poles of the inscribed triangles $A_1B_1C_1$ and $A_2B_2C_2$; ABC the original triangle, and $A_3B_3C_3$ the circumscribed triangle whose sides are parallel to those of $A_1B_1C_1$. Taking AC, AB for axes, let

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$$AC = \frac{1}{a},\ AB = \frac{1}{b},\ AB_1 = \frac{1}{m},\ AC_1 = \frac{1}{n},\ AB_2 = \frac{1}{m'},\ AC_2 = \frac{1}{n'};$$

then we have the following equations,

$$\begin{aligned}
&(AR) \text{ or } (B_3C_3) \ldots\ldots\ldots && mx + ny = 0\\
&\quad (B_2C_2) \ldots\ldots\ldots && m'x + n'y = 1\\
&(BQ) \text{ or } (A_3C_3) \ldots\ldots\ldots && n\,by + b\,(2a - m)\,x - n = 0\ *\\
&\quad (A_2C_2) \ldots\ldots\ldots && n'y + (2a - m')\,x - 1 = 0\\
&(CP) \text{ or } (A_3B_3) \ldots\ldots\ldots && max + a(2b - n)\,y - m = 0\\
&\quad (A_2B_2) \ldots\ldots\ldots && m'x + (2b - n')\,y - 1 = 0
\end{aligned}$$

Let $(A_3C_3) = \lambda\,(A_2C_2)$ and $(A_3B_3) = \mu\,(A_2B_2)$ ..... (a) and determine λ and μ so that the values of x and y deduced from (B_3C_3) and (B_2C_2) may satisfy those equations, and we find

$$\lambda = \frac{2b\,(a-m) - (m'n - mn')}{2\,(a-m')},$$

$$\mu = \frac{2a\,(b-n) - (m\,n' - m'n)}{2\,(b-n')};$$

and the two equations (a) reduce to one and the same, viz.

$$\begin{aligned}
&\{2ab\,(n-n') + (n' - 2b)\,(m'n - mn')\}y\\
&+ \{2ab(m-m') + (m' - 2a)\,(mn' - m'n)\}x\\
&\quad = 2\,(na + mb) - 2\,ab - (m'n - mn'),
\end{aligned}$$

which therefore passes through the three points of intersection of B_3C_3, B_2C_2; A_3C_3, A_2C_2; A_3B_3, A_2B_2.

[THIRD SOLUTION. *Mr. Rutherford.*]

Let ABC be a triangle, and $A_1B_1C_1$, $A_2B_2C_2$ two inscribed triangles, each being collinear and collinearly situated with respect to ABC. Since $A_1B_1C_1$ and $A_3B_3C_3$ are similar and similarly situated, by construction, they are therefore collinear and collinearly situated triangles; hence ABC and $A_3B_3C_3$ are collinear and collinearly situated triangles. Again $A_2B_2C_2$ is inscribed in ABC and collinear and collinearly situated with it, and ABC is similarly inscribed in $A_3B_3C_3$; hence $A_2B_2C_2$ and $A_3B_3C_3$ are collinear and collinearly situated triangles, and consequently their corresponding sides will necessarily intersect in the collinear axis, PQ, of these two systems. (See p. 192, Ex. 1.)

LXXV. *By P. K.*

Integrating by parts the formula $\int e^x\,dx = e^x$ it is required to show that

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + etc.$$

[FIRST SOLUTION. *Mr. R. Harley, Northwich, Cheshire; and H. R. L.*]

Integrating by parts we have

$$\int e^x\,dx = e^x.x - \int e^x x\,dx$$

$$\int e^x.x\,dx = \tfrac{1}{2}\,e^x.x^2 - \tfrac{1}{2}\int e^x x^2 dx$$

* This and the three subsequent equations are found in this manner. Find the equations of the lines CC_1 and BB_1, and then the equation of AA_1 which is the line passing through the origin A and the pole S or the point of intersection of CC_1 and BB_1. The equations of CB and AA_1 will give the coordinates of A_1; hence the equation of A_1C_1 is found, and finally the equation of BQ which is parallel to A_1C_1 and passes through the given point B.—EDS.

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$$\tfrac{1}{2}\int e^x.x^2dx=\frac{1}{2.3}e^x.x^3-\frac{1}{2.3}\int e^x x^3dx$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Hence $\int e^x dx$ or $e^x = e^x\left(x-\frac{1}{1.2}x^2+\frac{1}{1.2.3}x^3-etc.\right)+C.$

Dividing by e^x and transposing, we get

$$C.e^{-x}=1-x+\frac{1}{1.2}x^2-\frac{1}{1.2.3}x^3+etc.$$

If $x=0$; then $C=1$, and therefore

$$e^{-x}=1-x+\frac{1}{1.2}x^2-\frac{1}{1.2.3}x^3+etc.$$

Writing $-x$ for x, or which is the same thing, taking the reciprocals of both members of the equation, we obtain

$$e^x=1+x+\frac{1}{1.2}x^2+\frac{1}{1.2.3}x^3+etc.$$

[Second Solution. *Mr. Thomas Weddle; and similarly by Mr. Hearn.*]

Since $\frac{d}{dx}e^x=e^x$, and $\int x^n dx=\frac{x^{n+1}}{n+1}$, we have, integrating by parts

$$\int e^x x^n dx=\frac{1}{n+1}e^x x^{n+1}-\frac{1}{n+1}\int e^x x^{n+1}dx,$$

it being obvious that the integrals here vanish for $x=0$.

By a continuous application of this formula we obtain,

$$\begin{aligned}\int e^x dx &= e^x x-\int e^x x dx\\ &= e^x x-\tfrac{1}{2}e^x x^2+\tfrac{1}{2}\int e^x x^2 dx\\ &= e^x x-\tfrac{1}{2}e^x x^2+\frac{1}{2.3}e^x x^3-\frac{1}{2.3}\int e^x x^3 dx\\ &= e^x x-\tfrac{1}{2}e^x x^2+\frac{1}{2.3}e^x x^3-\frac{1}{2.3.4}e^x x^4+\frac{1}{2.3.4}\int e^x x^4 dx\\ &\ \ \vdots\\ &= e^x x-\tfrac{1}{2}e^x x^2+\frac{1}{2.3}e^x x^3-\frac{1}{2.3.4}e^x x^4+\ldots\ldots\ldots\ldots\ldots(1)\end{aligned}$$

Moreover in $\int e^x dx=e^x$, the integral vanishes for $x=-\infty$, if the inferior limit be changed to $x=0$, we must write $\int e^x dx=e^x-e^0$ or e^x-1; substitute this in (1), divide by e^x and transpose,

$$\therefore e^{-x}=1-x+\tfrac{1}{2}x^2-\frac{1}{2.3}x^3+\frac{1}{2.3.4}x^4+\ldots\ldots$$

Write $-x$ for x

$$\therefore e^x=1+x+\tfrac{1}{2}x^2+\frac{1}{2.3}x^3+\frac{1}{2.3.4}x^4+\ldots\ldots$$

the expansion required.

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LXXVI. *Mr. Fenwick.*

All surfaces of the second order described through the angular points of a given parallelopiped have the same centre, and the diameters of each drawn parallel to the edges of the parallelopiped are conjugate diameters.

[First Solution. *Mr. G. W. Hearn.*]

Taking a, b, c for the sides of the parallelopiped and for axes of reference, we have

$$\alpha x(x-a)+\beta y(y-b)+\gamma z(z-c)+hxy+kxz+lyz=0,$$

for the equation of the surface through four contiguous angles. In order that it may also pass through the other four, we must have simultaneously,

$x=a$	$x=a$	$x=0$	$x=a$
$y=b$	$y=b$	$y=b$	$y=0$
$z=c$	$z=0$	$z=c$	$z=c$

Hence $h=0$, $k=0$, $l=0$, and transferring the origin to a point $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$, we have

$$\alpha\left\{x^2-\frac{a^2}{4}\right\}+\beta\left\{y^2-\frac{b^2}{4}\right\}+\gamma\left\{z^2-\frac{c^2}{4}\right\}=0,$$

or
$$\alpha x^2+\beta y^2+\gamma z^2=\frac{\alpha a^2+\beta b^2+\gamma c^2}{4};$$

a surface concentric with the parallelopiped and having conjugate diameters parallel to its sides.

[Second Solution. *Mr. Thomas Weddle, and similarly by Messrs. S. Bills and James Reid.*]

Take the intersection of the diagonals for the origin, and the axes of x, y, and z parallel to the edges of the parallelopiped. Then the angular points of the parallelopiped may be denoted by $(x_1y_1z_1)$, $(-x_1-y_1-z_1)$, $(-x_1y_1z_1)$, $(x_1-y_1-z_1)$, $(x_1-y_1z_1)$, $(-x_1y_1-z_1)$, $(x_1y_1-z_1)$, $(-x_1-y_1z_1)$.

Let the equation of a surface of the second order described through the angular points of the parallelopiped be,

$$ax^2+by^2+cz^2+dyz+exz+fxy+gx+hy+kz+1=0,$$

or, for brevity,
$$\phi(x,y,z)=0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Hence we must have

$\phi(x_1\,y_1\,z_1)=0\ldots.(2)$, $\phi(-x_1-y_1-z_1)=0\ldots(3)$, $\phi(-x_1\,y_1\,z_1)=0\ldots(4)$

$\phi(x_1-y_1-z_1)=0\ldots(5)$, $\phi(x_1-y_1z_1)=0\ldots.(6)$, $\phi(-x_1y_1-z_1)=0\ldots(7)$

$\phi(x_1y_1-z_1)=0\ldots(8)$, $\phi(-x_1-y_1z_1)=0\ldots(9)$.

Take the sum of (3) and (4) from that of (2) (5),

$\therefore\ g=0$; similarly $h=0$, and $k=0$.

Also take the sum of (2, 3, 4, 5) from that of (6, 7, 8, 9),

$\therefore\ d=0$; similarly $e=0$, and $f=0$.

Hence each of the equations (2, 3....9) becomes,

$$ax_1^2+by_1^2+cz_1^2+1=0 \ldots\ldots\ldots\ldots\ldots (10),$$

and (1) reduces to
$$ax^2+by^2+cz^2+1=0 \ldots\ldots\ldots\ldots\ldots (11).$$

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As (11) is the equation of a surface of the second order referred to conjugate diameters, we infer that all the surfaces have the origin, or the intersection of the diagonals of the parallelopiped, for their centre, and the axes which are parallel to the edges of the parallelopiped, for conjugate diameters.

The equation (10) expresses the relation between a, b and c.

[THIRD SOLUTION. *Mr. J. W. Elliott, Greatham.*]

The general equation to a surface of the second order is

$$Ax^2+A'y^2+A''z^2+2Byz+2B'xz+2B''xy+2Cx+2C'y+2C''z+E=0\ldots(1)$$

Take three conterminous edges (a,b,c) of the given parallelopiped as the axes of coordinates: then the angular points will be denoted by

$$abc,\ abo,\ aoo,\ ooo,\ obc,\ ooc,\ obo,\ aoc.$$

Hence we determine in reference to (1),

$$2Bbc+E=0,\quad 2B'ac+E=0,\quad 2B''ab+E=0;$$

but $E=0$, since one point is at the origin, therefore $B=0$, $B'=0$, $B''=0$. Wherefore (1) becomes

$$Ax^2+A'y^2+A''z^2+2Cx+2C'y+2C''z=0\ldots\ldots\ldots\ldots(2)$$

Transform the origin of coordinates, by putting $x=\alpha+x_1$, $y=\beta+y_1$, $z=\gamma+z_1$, and this last equation becomes

$$Ax_1^2+A'y_1^2+A''z_1^2+2(A\alpha+C)x_1+2(A'\beta+C')y_1+2(A''\gamma+C'')z_1 +A\alpha^2+A'\beta^2+A''\gamma^2+2C\alpha+2C'\beta+2C''\gamma=0\ldots.(3)$$

Now the conditions, that the surface has a centre, are

$$A\alpha+C=0,\quad A'\beta+C'=0,\quad A''\gamma+C''=0\ldots\ldots\ldots(4)$$

but from (1), in reference to the points aoo, obo, ooc, we have

$$A\alpha+2C=0,\quad A'b+2C'=0,\quad A''c+2C'=0\ldots\ldots\ldots(5)$$

From (4) and (5) we obtain $\alpha=\frac{1}{2}a$, $\beta=\frac{1}{2}b$, $\gamma=\frac{1}{2}c$; hence the new origin is the *centre*; and observing (5), equation (3) becomes

$$Ax_1^2+A'y_1^2+A''z_1^2+\tfrac{1}{2}Ca+\tfrac{1}{2}C'b+\tfrac{1}{2}C''c=0,$$

which represents innumerable surfaces of the second order; and the diameters of each drawn parallel to the edges of the parallelopiped are conjugate.

LXXVII. *Mr. James Reid, Prior's Salford.*

If in each of the straight lines which form the edges of a tetrahedron any two points be taken equally distant from the middle points of the edges, the twelve points lie in a surface of the second order.

[SOLUTION. *Mr. G. W. Hearn.*]

I have shewn in my solution to the analogous plane problem proposed by Mr. Rutherford, that on making a conic section pass through four of the points in AB, AC, the coefficient of xy remains to be determined, and that on passing the curve through either of the points in BC, the same value of k is found.

Let a, b, c be the halves of three contiguous edges of the tetrahedron; m, n, p the distances of the points from the middle points of the edges.

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Then taking the edges for axes of reference, let there be a surface whose equation is

$$(b^2-n^2)(c^2-p^2)\{(x-a)^2-m^2\}+(c^2-p^2)(a^2-m^2)\{(y-b)^2-n^2\}$$
$$+(a^2-m^2)(b^2-n^2)\{(z-c)^2-p^2\}+kxy+k'xz+k''zy$$
$$=2(a^2-m^2)(b^2-n^2)(c^2-p^2).$$

When $x=0$, the coefficient of xy may be determined so that the curve may pass through the six points in the plane xy. When $x=0$, the coefficient of zy may be determined so that the curve may pass through the six points in zy; and when $y=0$, the coefficient of xz may be determined so that the curve may pass through the six points in xz. Hence the surface will pass through all the twelve points.

Mr. Weddle gives a good solution, shewing that the twelve points in general lie in a *surface of the second order*.

LXXVIII. *Mr. J. F. Heather, M. A., R. M. Academy.*

An elastic ring being placed upon a sphere which revolves round a vertical axis, and the diameters of the ring and sphere being given, determine the angular velocity that the ring may slip completely over the sphere.

[First Solution. *Mr. G. W. Hearn, R. M. College.*]

Let ω be the angular velocity of the sphere, and a its radius, a the radius of the ring unstretched, r its radius under the tension t, and e the modulus of elasticity: then $r=a(1+et)$. The forces keeping in equilibrium an element $r\delta\theta$ of the ring (W being the whole weight of the ring) are

1st Weight of element . . . $=\frac{W\delta\theta}{2\pi}$,

2nd Normal pressure . . . $=\frac{N\delta\theta}{2\pi}$,

3rd Friction . . . . . $=\frac{\mu N\delta\theta}{2\pi}$, *

4th Pressure produced by tension $=t\delta\theta$,

5th Centrifugal force . . . $=r\omega^2.\frac{W\delta\theta}{2\pi g}$.

Hence if ϕ be the angle between a vertical and the radius of the sphere drawn to the circumference of the ring in a position of equilibrium, the equations of equilibrium will be

$$N(\sin\phi-\mu\cos\phi)+W\frac{r\omega^2}{g}=2\pi t$$

$$W-N(\cos\phi+\mu\sin\phi)=0.$$

Now since $r=a\sin\phi$, therefore $a\sin\phi=a(1+et)$; hence

$t=\frac{a\sin\phi-a}{ae}$; consequently the preceding equations give

$$\frac{W}{2\pi}\left\{\frac{\sin\phi-\mu\cos\phi}{\cos\phi+\mu\sin\phi}+\frac{a\omega^2}{g}\sin\phi\right\}=\frac{a\sin\phi-a}{ae},$$

* Friction *must* be considered, for if it did not exist no angular velocity would be communicated to the ring.

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for determining ϕ. If $\phi = \frac{\pi}{2}$ be one of the possible values of ϕ, we have

$$\omega^2 = \frac{2\pi g}{e\mathrm{W}} \frac{a-a}{aa} - \frac{g}{\mu a},$$

and if ω^2 in the slightest degree exceed this value, the ring will slip over the equator, and fall to the ground.

[SECOND SOLUTION. *Mr. J. F. Heather, the Proposer; and similarly by H. R. L.*]

Let R = radius of the sphere, r = radius of ring, m = unit of mass of ring when stretched, λ = elasticity of extension, t = tension, and ω = required angular velocity; then

$$\mathrm{R} = r(1+\lambda t),$$

$$\text{and} \qquad 2t\sin\frac{a}{2} = \frac{\mathrm{R}\omega^2}{g} . ma,$$

where a = an arc of the ring, and a = the angle subtended by arc a at the centre of the ring; hence we have

$$t = \frac{\mathrm{R}\omega^2}{2g} . \frac{ma}{\sin\frac{1}{2}a}.$$

Now when the arc is taken infinitely small, $\frac{a}{\sin\frac{1}{2}a} = 2\mathrm{R}$; hence

$$t = \frac{\mathrm{R}^2\omega^2}{g} . m;$$

$$\therefore \ \mathrm{R} = r\left(1 + \lambda\frac{\mathrm{R}^2\omega^2 m}{g}\right);$$

from which we obtain (W being the weight of ring)

$$\omega^2 = \frac{g}{\lambda \mathrm{R}m}\left\{\frac{1}{r} - \frac{1}{\mathrm{R}}\right\} = \frac{2\pi g}{\lambda \mathrm{W}}\left\{\frac{1}{r} - \frac{1}{\mathrm{R}}\right\}.$$

*** In this solution the angular velocity is supposed to be communicated, not by friction, but by some other force, and the inquiry is simply,—what angular velocity must be communicated to the elastic ring that its radius may equal that of the given sphere.—EDS.

LXXIX. *Professor Hearn, R. M. College, Sandhurst.*

On a rough board inclined at a given angle to the horizon a rough ball is projected horizontally: required its path and a confirmation of the result from actual experiment.

[SOLUTION. *Mr. Hearn, the Proposer.*]

The following principles are assumed as the basis of calculation.

1st. That a ball may describe any path *by rolling* on a plane, provided the axis of instantaneous rotation passes through the centre of curvature of the path at every instant.

2nd. That when the dimensions of the ball are small compared with the radius of curvature of the path, the axis of instantaneous rotation may without sensible error be regarded as parallel to the plane and at right angles to the path.

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3rd. That the force of friction causing the rotation always acts in a direction contrary to the actual motion, and therefore tangentially to the path.

4th. The roughness of the ball and plane are supposed to be sufficient to prevent sliding.

Let a be the radius of the ball, k its radius of gyration; $\mu = \frac{k^2}{a^2}$: s the length of the path measured from the point of projection; x the horizontal abscissa: and y the ordinate measured *down* the plane; M the mass of the ball; i the inclination of the plane.

The angular velocity of the ball round an axis at right angles to the path is $\frac{1}{a}\cdot\frac{ds}{dt}$, and $\therefore$ the angular accelerating force $\frac{1}{a}\cdot\frac{d^2s}{dt^2} = \frac{Fa}{Mk^2}$ $\therefore$ $\frac{F}{M} = \mu\frac{d^2s}{dt^2}$.

The equations of motion are

$$\frac{d^2x}{dt^2} + \mu\frac{d^2s}{dt^2}\cdot\frac{dx}{ds} = 0 \quad \ldots\ldots(1)$$

$$\frac{d^2y}{dt^2} + \mu\frac{d^2s}{dt^2}\cdot\frac{dy}{ds} = g\sin i \quad \ldots\ldots(2)$$

The integral of (1) is

$$\frac{dx}{dt}\left(\frac{ds}{dt}\right)^{\mu} = V^{1+\mu} \quad \ldots\ldots(3)$$

where V is the initial horizontal velocity.

Multiplying (1) by $\frac{dx}{dt}$, (2) by $\frac{dy}{dt}$, and adding

$$(1+\mu)\frac{d^2s}{dt^2}\cdot\frac{ds}{dt} = g\sin i\frac{dy}{dt}.$$

Integrating $(1+\mu)\frac{ds^2}{dt^2} = 2f(y+h)$ $\ldots\ldots(4)$

where $f = g\sin i$, and $h = \frac{(1+\mu)V^2}{2f}$, or H being the height due to V, $V^2 = 2gH$, and $\therefore$ $h = (1+\mu)H\operatorname{cosec} i$ $\ldots\ldots(5)$

The equation (4) may also be written

$$M(v^2 - V^2) = M\frac{2g\sin i}{1+\mu}\cdot y \quad \ldots\ldots(6)$$

under which form it expresses the gain of vis viva of motion of translation.

The equation (3) gives

$$\frac{ds}{dx} = \left(\frac{1}{V}\cdot\frac{ds}{dt}\right)^{1+\mu}$$

$$= \left(\frac{1}{V}\right)^{1+\mu}\left\{\frac{2f(y+h)}{1+\mu}\right\}^{\frac{1+\mu}{2}},$$

or $$\frac{ds^2}{dx^2} = \left(1+\frac{y}{h}\right)^{1+\mu} = 1+\frac{dy^2}{dx^2}$$

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$$\therefore\ 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = \frac{\mu+1}{h}\left(1+\frac{y}{h}\right)^{\mu}\cdot\frac{dy}{dx},$$

$$\frac{d^2y}{dx^2} = \frac{\mu+1}{2h}\left(1+\frac{y}{h}\right)^{\mu},$$

$$\frac{d^3y}{dx^3} = \frac{\mu(\mu+1)}{2h^2}\left(1+\frac{y}{h}\right)^{\mu-1}\frac{dy}{dx},$$

$$\frac{d^4y}{dx^4} = \frac{\mu(\mu+1)}{2h^2}\left(1+\frac{y}{h}\right)^{\mu-1}\frac{d^2y}{dx^2} + etc.;$$

from which we deduce when $y=0$, $x=0$

$$\left(\frac{dy}{dx}\right)_0 = 0;\ \left(\frac{d^2y}{dx^2}\right)_0 = \frac{\mu+1}{2h};\ \left(\frac{d^3y}{dx^3}\right)_0 = 0;$$

$$\left(\frac{d^4y}{dx^4}\right)_0 = \frac{\mu(\mu+1)^2}{4h^3},\ etc.;$$

and therefore by Maclaurin's series

$$y = \frac{\mu+1}{2h}\cdot\frac{x^2}{2} + \frac{\mu(\mu+1)^2}{4h^3}\cdot\frac{x^4}{2.3.4} + etc.;$$

or $$4\mathrm{H}\operatorname{cosec} i.y = x^2\left\{1+\frac{\mu}{24(\mu+1)}\cdot\left(\frac{x\sin i}{\mathrm{H}}\right)^2 + etc.\right\}.$$

When the inclination i is small the series within brackets converges very rapidly so long as $x \lessdot \mathrm{H}\operatorname{cosec} i$.

For a solid sphere $\mu = \frac{2}{5}$ $\therefore\ \frac{\mu}{24(\mu+1)} = \frac{1}{84}$ and therefore for moderate values of x the second term is scarcely sensible.

Hence within moderate limits of x the path is

$$x^2 = 4\mathrm{H}\operatorname{cosec} i.y,$$

which is the same as when there is no friction.

Hence we conclude as follows, that the actual path has an osculation of the third order with the parabolic path described, cæteris paribus, on a smooth plane, and *sensibly* coincides with it for a considerable extent, but that afterwards the horizontal range in the path *with* friction falls sensibly short of the range in the path *without* friction. But although the form of a part of the path described is *sensibly* the same in both cases, the velocity with which it is described will be *sensibly* different. Thus on reference to (6) we see that gain of vis viva of motion of translation without friction is to its gain with friction as $1+\mu : 1$ or $7 : 5$.

All these conclusions I have fully verified by experiment, and my object in proposing the problem was to call the attention of students to a mode of investigation in cases of the kind pursued by Mr. Earnshaw and subsequently by Mr. Walton.

See Earnshaw's Dynamics, p. 280, 3d edition, and Walton's Mechanical Problems, p. 444. The problem proposed in those works is as follows:— A circular board has a motion of rotation round a vertical axis passing through its centre perpendicular to its plane, it moves with uniform angular velocity, and a ball is laid upon it; supposing friction to prevent sliding, it is required to shew that the path of the ball in free space is *a circle concentric with the centre of the board.*

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This has always seemed to me to be a strange result, and quite at variance with experience; but it will be seen that the authors to whom I refer produce a still stranger result; namely, that with certain initial conditions the path may be a circle *not* concentric with the centre of the board.

I will here give my opinion concerning their mode of investigation.

The friction is resolved in the directions of x and y and taken proportional to $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$.

Now $\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 = \left(\frac{d^2s}{dt^2}\right)^2 + \left(\frac{v^2}{\rho}\right)^2$.

This mode of investigation therefore comes to this,—That the friction acts *obliquely* to the path, and may be resolved into $\mu \frac{d^2s}{dt^2}$ tangential to the path, and $\mu \frac{v^2}{\rho}$ perpendicular to it, v being the velocity, and ρ the radius of curvature. But friction and every other resisting force always acts contrary to the motion, and therefore *tangentially* to the path described. It seems therefore to be explained how these authors have come to such strange conclusions. If their method be applied to the problem here proposed, the equations of motion are

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} + \mu \frac{d^2y}{dt^2} = g \sin i,$$

and the path described is the parabola whose equation is

$$x^2 = 4(\mu + 1) \text{ H cosec } i.y,$$

which makes the horizontal range *with* friction greater than that *without* it, in the ratio of $\sqrt{1+\mu} : 1$ or $\sqrt{7} : \sqrt{5}$. I am satisfied that *this* is not borne out by experiment.

I will merely add, that I have treated Mr. Earnshaw's problem by the method pursued in this paper, and have found in general a spiral path receding from the centre of motion, and that for certain initial conditions both the relative and absolute paths are *logarithmic* spirals.

LXXX. *By Juvenis.*

A merchant bought a cask of spirits for £48, and sold a quantity exceeding three-fourths of the whole by two gallons at a profit of £25 per cent. He afterwards sold the remainder at such a price as to clear £60 per cent by the whole transaction; and had he sold the whole quantity at the latter price he would have gained £175 per cent: find the number of gallons contained in the cask.

[First Solution. *Henri Mondeux, the Shepherd of Touraine.*]

*** This wonderful French youth having lately visited Jersey to exhibit his extraordinary powers of mental calculation, the above exercise was proposed to him, and a nearly instantaneous answer was given. At the request of our valued correspondent, Mr. Godfray, he wrote out his solution, of which the following translation has been sent us by that gentleman.

He sells the first portion at a profit of £25 per cent, and the last at £175 per cent, and gains £60 per cent on the whole. The first profit is less than the mean profit by £35 per cent, and the second is greater by £115 per cent; he has therefore sold 115 parts of the first against 35 of the second; that is, the first portion sold was the $\frac{115}{150}$ of the whole cask, and the last the

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$\frac{35}{150}$; but the first portion was $\frac{3}{4}$ of the cask, and 2 gallons more, and the difference between $\frac{115}{150}$ or $\frac{23}{30}$ and $\frac{3}{4}$ is $\frac{1}{60}$; therefore 2 gallons is the $\frac{1}{60}$ of the cask; that is, the cask contained 120 gallons.

The solution of Mr. James Reid was analogous to the above.

[Second Solution. *Mr. Thomas Wilkinson, Cheapside, Burnley.*]

Here, as 100 : 60 :: £48 : £28$\frac{4}{5}$ = the gain on the whole transaction.

Again, 100 : 25 :: $\frac{3}{4}$ of £48 : £9 = the gain on $\frac{3}{4}$ of the whole.

And 100 : 25 :: 2 galls. : $\frac{1}{2}$ gal. = the gain on 2 gallons.

Also 100 : 175 :: $\frac{1}{4}$ of £48 : £21 = the gain on $\frac{1}{4}$ of the whole at the latter price.

And 100 : 175 :: 2 galls. : 3$\frac{1}{2}$ galls. = the gain on the 2 galls. at the latter rate, therefore the whole gain is £9+£21+$\frac{1}{2}$ gall.—3$\frac{1}{2}$ gall., which is = £28$\frac{4}{5}$.

Consequently £30—£28$\frac{4}{5}$ = £1$\frac{1}{5}$ is the price of 3 gallons.

Hence, £1$\frac{1}{5}$: £48 :: 3 gall : 120 gallons, the content of the cask.

[Third Solution. *William Burns, L.L.D., Rochester, and Mr. G. W. Hearn.*]

As 100 : 175 :: £48 : £84; hence £84 divided by 4 or £21 is the gain on one-fourth of the number of gallons, if sold at £175 per cent profit.

As 100 : 25 :: £48 : £12; hence three-fourths of £12 is £9, the gain on three-fourths of the number of gallons, if sold at £25 per cent profit; therefore the whole gain would have been £30.

But 100 : 60 :: £48 : £28·8, the actual gain; hence £1·2 is the difference of gain on 2 gallons at £175 and 2 at £25 per cent, or the gain on 2 gallons at £150 per cent profit.

Lastly, 100 : 150 :: £48 : £72, the gain on the whole at £150 per cent., therefore £ 1·2 : £ 72 : : 2 gallons : 120 gallons, the number in the cask.

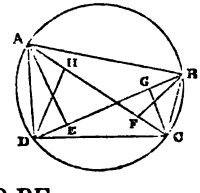
Solutions were received from Messrs. W. H. Levy, Shalbourne; John Duxbury, Mechanic's Institution, Blackburn; R. Harley, Northwich; and A. D. B., Charlton Cottage, near Blandford.

LXXXI. *Mr. R. Harley, Northwich.*

From each angle of a quadrilateral ABCD inscribed in a circle let a line be inflected to the opposite diagonal, so that it and the other diagonal may make equal angles with the adjacent sides: then will the areas of the triangles ABC, ADC, ABD, BCD, into which the quadrilateral is divided by its diagonals, be to each other respectively as the four inflected lines.

[First Solution. *Messrs. Hearn, Reid, and Weddle; and similarly by A. D. B., Charlton Cottage, near Blandford, Dorset; Mr. W. H. Levy, Shalbourne; and Dr. Burns, Rochester.*]

Let AE, BF, CG and DH be the lines inflected to the diagonals.



Since ∠ABF = (by hyp.) ∠DBC, and ∠BAF = ∠BDC, in the same segment, the triangles ABF and BDC are equiangular; in the same way, it is shown that the triangles ABC and DGC are also equiangular.

Since the triangles ABF and BDC are equiangular,

AB : BF :: BD : BC; ∴ AB·BC = BD·BF.

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Moreover, the triangles ABC and DGC have the angles at B and G equal,

$$\therefore \Delta ABC : \Delta DGC :: AB.BC : DG.GC$$
$$:: BD.BF : DG.GC.$$

Also $$\Delta DGC : \Delta BDC :: \text{base } DG : \text{base } BD$$
$$:: DG.GC : BD.GC.$$

Hence, by equality,

$$\Delta ABC : \Delta BDC :: BD.BF : BD.GC$$
$$:: BF : GC.$$

The other analogies are established in a similar way.

[Second Solution. *Mr. J. W. Elliott, Greatham, and Mr. Harley, the Proposer.*]

Let AE, BF, CG, DH be the inflected lines, and put $AB = a$, $BC = b$, $CD = c$, $DA = d$, $BD = \alpha$, $AC = \beta$, $AE = a'$, $BF = b'$, $CG = c'$, $DH = d'$. Denote the areas of the triangles ABC, ADC, ABD, BCD by $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, respectively; and the diameter of the circle circumscribed to the quadrilateral, by D. Then from Euc. VI. C. and similar triangles, we have the following equations:—

$$\left.\begin{aligned} 2D\Delta_1 &= ab\beta \\ 2D\Delta_2 &= cd\beta \\ 2D\Delta_3 &= ad\alpha \\ 2D\Delta_4 &= bc\alpha \end{aligned}\right\} \ldots\ldots (1) \qquad \left.\begin{aligned} ab &= b'a \\ cd &= d'a \\ ad &= a'\beta \\ bc &= c'\beta \end{aligned}\right\} \ldots\ldots (2)$$

Inserting the equations marked (2) in (1), we get

$$\left.\begin{aligned} 2D\Delta_1 &= a\beta b' \\ 2D\Delta_2 &= a\beta d' \end{aligned}\right\} \ldots\ldots (3) \qquad \left.\begin{aligned} 2D\Delta_3 &= a\beta a' \\ 2D\Delta_4 &= a\beta c' \end{aligned}\right\} \ldots\ldots (4)$$

Consequently, $$\frac{a\beta}{2D} = \frac{\Delta_1}{b'} = \frac{\Delta_2}{d'} = \frac{\Delta_3}{a'} = \frac{\Delta_4}{c'},$$

which proves the property enunciated.

Cor. 1. From (1), $2D(\Delta_1+\Delta_2) = \beta(ab+cd)$, and $2D(\Delta_3+\Delta_4) = \alpha(ad+bc)$; hence, since $\Delta_1+\Delta_2 = \Delta_3+\Delta_4$,

$$\frac{\alpha}{\beta} = \frac{ab+cd}{ad+bc}, \text{ or } \frac{BD}{AC} = \frac{AB.BC + CD.DA}{AB.AD + BC.CD}.$$

This is a well known property.

Cor. 2. By (3, 4),

$$2D(\Delta_1+\Delta_2) = a\beta(b'+d'), \text{ and } 2D(\Delta_3+\Delta_4) = a\beta(a'+c');$$

hence, $a'+c' = b'+d'$, or $AE+CG = BF+DH$. (See Diary, Quest. 1755.)

Cor. 3. By (2), $$abcd = a^2b'd' = \beta^2a'c',$$

$$\therefore \frac{a^2}{\beta^2} = \frac{a'c'}{b'd'}.$$

That is, *the rectangles of the lines inflected to the opposite diagonals are to each other as the squares of those diagonals.*

This property is due to Mr. Elliott.

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[THIRD SOLUTION. *Mr. W. Mortimer, Seedley, near Manchester.*]

Let AE, BF be two of the lines described in the question. Then in the triangles ABE, ACD, the angle ABE = ACD (Euc. III. 21), and BAE = CAD (by hypothesis), therefore these triangles are equiangular.

Hence AE : AB :: AD : AC, whence AE.AC = AB.AD.

Similarly, BF.BD = AB.BC. Consequently,

$$AE : BF :: \frac{AB.AD}{AC} : \frac{AB.BC}{BD} :: AB.AD.BD : AB.BC.AC.$$

But AB.AD.BD : AB.BC.AC :: 4R.ΔABD : 4R.ΔABC.

Hence $\Delta ABD : \Delta AEC :: AE : BF$.

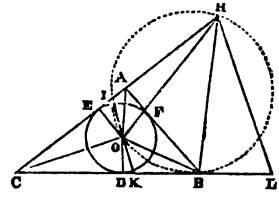
Good solutions were also received from Ambulator and Mr. Wright, London.

LXXXII. *By Geometricus.*

Given the sum of the two sides of a plane triangle, the radius of the inscribed circle, and either of the angles at the base, to construct the triangle.

[FIRST SOLUTION. *Mr. Hugh Godfray, David Place, St. Mark's Road, Jersey.*]

Let DEF be the given circle touching the line CB in D; draw the radius OD, and at the point O make the angle DOE equal to the supplement of the given angle. Through E draw the tangent line CH intersecting CB in C: take CH equal to the given sum of the sides, and draw the lines CO, OH and also KOI at right angles to the bisecting line CO cutting CH in I. Describe a circle through H, I, O determining the point B in BC. Draw the tangent BA, and ABC will be the triangle required.



For draw OB and OA; then the three angles COB, OCB, OBC are together equal to two right angles; and the angles OAB, OBC, OCB are obviously together equal to a right angle; hence OAB together with a right angle are equal to COB; but COB is equal to COK and KOB, that is, COB is equal to a right angle together with AHB (since AHB is equal to KOB); therefore OAB or OAC is equal to AHB; hence HB is parallel to AO, and therefore ABH is equal to AHB, and AB to AH: consequently CA and AB are together equal to CH, the given sum of the sides. There will be two, one or no solutions according as the circle HIO cuts the line CB in two, one or no points.

[SECOND SOLUTION. *W. Burns, L.L. D., Classical, Commercial. and Mathematical School, St. Margaret's Bank, Rochester; and in a similar manner by Mr. R. Harley, Northwich, Cheshire.*]

Describe the given circle DEF touching the line CB in D, and let O be its centre; draw OD and make the angle DOE equal to the supplement of the given angle at the base, and through E draw a line CH touching the circle and intersecting CB in C: take CH and CL, each equal to the sum of the sides, and draw HL and HO. On HO describe a segment of a circle containing an angle equal to the angle CLH and let it intersect CL in B: draw BA touching the circle DEF in F, and ABC is the triangle required.

For draw OB, OC, and BH,; then since BD and BF are tangents to the circle, and OB drawn from the centre, the angle OBD is equal to OBF; also CBH is equal to BLH and BHL together, and OBH is, by construc-

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tion, equal to CLH; therefore OBD or OBF is equal to BHL. But since CH is equal to CL, angle CHL is equal to CLH or OBH; therefore ABH is obviously equal to AHB, and AH to AB; hence CA, AB are together equal to the given sum of the sides. The construction is hence obvious.

When the circle described on OH cuts CL in two points; two triangles may be constructed answering the conditions; and if the circle touch CL the two triangles will unite in one, and if the circle does not meet CL, then no triangle can be constructed with the data.

Mr. Thomas Weddle and Ambulator sent geometrical solutions of this exercise. Messrs Hearn, Levy, Mortimer, and A. D. B. favoured us with excellent analytical solutions.

LXXXIII. *Mr. James Reid, Prior's Salford.*

If from the point in which a normal to an ellipse meets the axis major a straight line be drawn at right angles to the normal, the part of either of the focal distances intercepted between this line and the curve will be an harmonic mean between the focal distances.

[First Solution. *Dr. Burns, Rochester; and Mr. Thomas Weddle, Newcastle.*]

Let the normal PN at the point P meet the major axis AB in N, and let QNQ′ be drawn perpendicular to PN meeting the focal distances FP, F′P in Q and Q′; also let the tangent at P meet AB produced in M.

Then since FP, F′P make equal angles with the tangent, and NPM, PNQ′ are right angles, QPN is equal to Q′PN, QN to NQ′, and PQ to PQ′. Moreover, the lines PM, QN being parallel,

$$FP : FQ :: PM : QN :: PM : NQ';$$

and the triangles PF′M, NF′Q′ being similar,

$$PM : NQ' :: PF' : F'Q'. \text{ Hence}$$

$$FP : FQ :: F'P : F'Q';$$

or since

$$PQ = PQ',$$

$$F'P : FP :: F'P - PQ : PQ - FP.$$

That is, PQ or PQ′ is a harmonic mean between F′P and FP.

[Second Solution. *Mr. G. W. Hearn, R. M. College.*]

Let F and F′ be the foci, PN the normal, which bisects the angle FPF′ and meets the major axis in N. Make PF axis of x, PF′ axis of y, and draw QNQ′ perpendicular to PN meeting PF, PF′ in Q and Q′; then $PQ = PQ' = a$. The equation to PN is $y - x = 0$, and that to QQ′ is $y + x = a$; hence

$$y + x + m(y - x) = a,$$

is the equation to FF′.

Make $x = 0$, then $y = \frac{a}{1+m} = PF'$.

.... $y = 0$, ...$x = \frac{a}{1-m} = PF$.

Hence $$\frac{1}{PF'} + \frac{1}{PF} = \frac{2}{a} = \frac{2}{PQ} \text{ or } \frac{2}{PQ'},$$

PQ or PQ′ is therefore a harmonic mean between PF′ and PF.

Correct and elegant solutions were also received from Messrs. J. W. Elliott, Greatham; R. Harley, Northwich; W. Mortimer, Seedley, near Manchester; John Taylor, Woolwich Common; and R. H. Wright, London.

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LXXXIV. *Mr. John Riddle, Nautical School, Greenwich Hospital.*

A statue a feet high placed upon a column subtended an angle of $\theta°$ at a certain distance from the base, and b feet farther off it subtended an angle of $\phi°$; find the height of the column and distance of the observer from it at first.

[First Solution. *Messrs. J. W. Elliott; R. Harley; G. W. Hearn; H. Hewett, Mr. Paul's School, Shooter's Hill; James Reid; John Riddle, the Proposer; T. Weddle; R. H. Wright; and John Taylor, Dr. Bridgman's School, Woolwich Common.*]

Let x = PB, the distance of first station from the base of the column,
y = PO, the height of the column,
θ' = angle of elevation PBO of the top of the column at first station,
ϕ' = the corresponding angle PEO at the second station.

Then
$$\frac{y}{x} = \tan\theta', \quad \frac{y}{x+b} = \tan\phi' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{y+a}{x} = \tan(\theta'+\theta) = \frac{\tan\theta'+\tan\theta}{1-\tan\theta'\tan\theta} \ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{y+a}{x+b} = \tan(\phi'+\phi) = \frac{\tan\phi'+\tan\phi}{1-\tan\phi'\tan\phi} \ldots\ldots\ldots\ldots\ldots (3)$$

The values of $\tan\theta'$ and $\tan\phi'$ given in (1) being substituted in (2) and (3), we have

$$y^2+ay+x^2 = ax\cot\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$y^2+ay+x^2+2bx+b^2 = a(x+b)\cot\phi \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

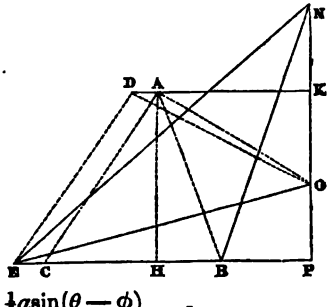
Deduct (4) from (5), and the resulting equation gives

$$x = \frac{b(a\cot\phi - b)}{2b+a(\cot\theta - \cot\phi)}.$$

The value of x being substituted in (4), the value of y may be found by the solution of a quadratic equation.

[Second Solution. *Mr. H. Godfray; and similarly by Ambulator, W. Burns, L. L. D., and Master W. Mortimer, Seedley, near Manchester.*]

Let NO be the statue on the column PO; B and E the two positions of the observer; let A and D be the centres of the two circles passing through N, O, B and N, O, E respectively. Then DAK bisects NO at right angles; draw AC parallel to DE; AH perpendicular to PE, and draw AO, DO.



Now it is obvious that angle KAO = θ, and angle KDO = ϕ; hence

$$AB = AO = \tfrac{1}{2}\, a\,\mathrm{cosec}\theta,$$

$$AC = DE \text{ or } DO = \tfrac{1}{2}\, a\,\mathrm{cosec}\phi,$$

$$DA = DK - AK = \tfrac{1}{2}\, a(\cot\phi - \cot\theta) = \frac{\frac{1}{2}a\sin(\theta-\phi)}{\sin\theta\sin\phi} = d,$$

$$CB = BE - DA = b - d.$$

The three sides of the triangle ABC being thus determined, the angle B can be found, and thence

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$$PO = AH - KO = AB\sin B - \tfrac{1}{2}a = \frac{\tfrac{1}{2}a(\sin B - \sin\theta)}{\sin\theta}$$
$$= \frac{a\sin\tfrac{1}{2}(B-\theta)\cos\tfrac{1}{2}(B+\theta)}{\sin\theta},$$

and $$PB = KA - BH = \tfrac{1}{2}a\cot\theta - AB\cos B = \frac{\tfrac{1}{2}a(\cos\theta - \cos B)}{\sin\theta}$$
$$= \frac{a\sin\tfrac{1}{2}(B-\theta)\sin\tfrac{1}{2}(B+\theta)}{\sin\theta}.$$

A. D. B., of Charlton Cottage, near Blandford, Dorset, favoured us with a solution.

LXXXV. *By Penumbra.*

There are given three lines SP, SQ, SR, and two points A, B, all in the same plane. Let any triangle PQR be described, having its angular points upon the three given lines, and two of its sides passing through the given points; then, whatever be this triangle, its third side will pass through a given point.

[First Solution. *Pen and Ink; and similarly by Mr. R. Harley.*]

Let the lines and points of the following figure represent those specified in the enunciation; draw the line AB to meet SP in D, SQ in E, SR in F, and PQ in C.

Then the triangles ARB, BRF, FRA being respectively cut by the transversals PQ, QS, SP, we have (*Hutton*, ii. p. 215.)

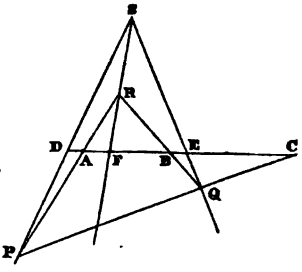
$$RP.AC.BQ = PA.CB.QR,$$
$$QR.SF.EB = BQ.RS.FE,$$
$$AP.RS.FD = PR.SF.DA;$$

and hence compounding, we obtain

$$AC.FD.EB = CB.DA.FE.$$

The ratio AC : CB is therefore given, and hence C is a given point in AB.

The same demonstration applies whatever be the relative positions of the lines and points.



Mr. Harley sent two other good solutions to this exercise.

[Second Solution. *Mr. G. W. Hearn.*]

Take S for origin, SR for axis of x, SQ for axis of y; and let

$u = ax+by+1 = 0$, be equation to AB
$y = ax$ ..... SA
$y = \beta x$ ..... SB

Let λ, μ be two arbitrary multipliers: then

(1) $y - ax + \lambda u = 0$ is any line through A
(2) $y - \beta x + \mu u = 0$ ..... B

Whence the line SP becomes

$$\mu(y - ax) - \lambda(y - \beta x) = 0.$$

But this line is invariable (by the hypothesis) and hence $\frac{y}{x}$ is constant: that is $\frac{\mu a - \lambda\beta}{\mu - \lambda}$ is constant; or, again, $\frac{\lambda}{\mu}$ is constant.

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Put $y = 0$ in (1), and hence $x = \frac{\lambda}{a - \lambda a} = \text{SR}$,

$x = 0$ in (2), ... $y = -\frac{\mu}{1 + \mu b} = \text{SQ}$.

Now the equation of the line QR through the two points just determined, is,

$$\frac{a - \lambda a}{\lambda} x - \frac{1 + \mu b}{\mu} y = 1, \text{ or,}$$

$$(3) \qquad y - \frac{\mu a}{\lambda} x + \mu u = 0 :$$

which, obviously, expresses a line drawn through the intersection of

$$u = 0 \text{ and } y - \frac{\mu a}{\lambda} x = 0.$$

But $\frac{\mu}{\lambda}$ is constant; and therefore both the lines are invariable, and their mutual intersection is hence a fixed point.

Solutions were also sent by Messrs. Elliott, Reid, Weddle, and Wright. Mr. Weddle's solution is very elegant, and is omitted with much regret.

LXXXVI. *By Amicus.*

In every tetrahedron, the sum of the squares of the six edges is equal to four times the sum of the squares of the lines which join the middles of the opposite edges.

[First Solution. *Mr. J. W. Elliott, Greatham.*]

Let ABCD be the tetrahedron, and G, H, K, L, M, N, the middle points of the edges AC, AB, BD, BC, DA, DC respectively.

Then, since the line drawn to bisect two sides of a triangle is parallel to the third side, and equal to half of it, we have

$$\text{BC} = 2\text{KN} = 2\text{HG}, \qquad \text{DC} = 2\text{MG} = 2\text{LK},$$
$$\text{AC} = 2\text{HL} = 2\text{MN}, \qquad \text{DA} = 2\text{KH} = 2\text{GN},$$
$$\text{AB} = 2\text{LG} = 2\text{KM}, \qquad \text{BD} = 2\text{HM} = 2\text{LN}.$$

Now since KN and HG are parallel to BC they are parallel to one another (*Euc. xi.* 9); and each being the half of BC, they are also equal to one another. Wherefore KNGH is a parallelogram, the diagonals of which are GK, HN; and hence by a known property,

$$2\text{GK}^2 + 2\text{HN}^2 = 4\text{HK}^2 + 4\text{KN}^2 = \text{AD}^2 + \text{BC}^2.$$

Taking in the same way the parallelograms MNHL and MKGL, and writing the entire results, we have

$$2\text{GK}^2 + 2\text{HN}^2 = \text{AD}^2 + \text{BC}^2,$$
$$2\text{HN}^2 + 2\text{ML}^2 = \text{BD}^2 + \text{AC}^2,$$
$$2\text{ML}^2 + 2\text{GK}^2 = \text{AB}^2 + \text{DC}^2.$$

These being added give the result stated in the theorem.

Corollary. The three diagonals GK, HN, LM meet in one point O.

This is easily proved to be the centre of gravity of the tetrahedron.

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[Second Solution. *Mr. Wright, London.*]

If the given figure were regular, the solution would be very simple; but as this is not so, let the points A, B, C, D, be represented by

$$x_1y_1x_1,\ x_2y_2x_2,\ x_3y_3x_3, \text{ and } ooo.$$

Also put

$\Sigma(\delta^2)$ = sum of the squares of the edges;

$\Sigma(\Delta^2)$= ..... lines which join the middle points of the opposite edges;

$\Sigma(x^2y^2z^2)$= ..... all the coordinates;

$\Sigma(x,y,z)$ = sum of all the rectangles $x_1x_2,\ x_2x_3,\ x_3x_1,\ y_1y_2 \ldots z_3z_1$.

Then we have

$$\begin{aligned} AD^2 &= x_1^2+y_1^2+z_1^2, \\ BD^2 &= x_2^2+y_2^2+z_2^2, \\ CD^2 &= x_3^2+y_3^2+z_3^2, \end{aligned} \quad \left| \quad \begin{aligned} AB^2 &= (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2, \\ BC^2 &= (x_2-x_3)^2+(y_2-y_3)^2+(z_2-z_3)^2, \\ CA^2 &= (x_3-x_1)^2+(y_3-y_1)^2+(z_3-z_1)^2. \end{aligned} \right.$$

Whence,

$$\Sigma(\delta^2) = 3\Sigma(x^2y^2z^2) - 2\Sigma(x,y,z).$$

Again,

$$ML^2=\left(\frac{x_1}{2}-\frac{x_2+x_3}{2}\right)^2+\left(\frac{y_1}{2}-\frac{y_2+y_3}{2}\right)^2+\left(\frac{z_1}{2}-\frac{z_2+z_3}{2}\right)^2,$$

$$KG^2=\left(\frac{x_2}{2}-\frac{x_3+x_1}{2}\right)^2+\left(\frac{y_2}{2}-\frac{y_3+y_1}{2}\right)^2+\left(\frac{z_2}{2}-\frac{z_3+z_1}{2}\right)^2,$$

$$HN^2=\left(\frac{x_3}{2}-\frac{x_1+x_2}{2}\right)^2+\left(\frac{y_3}{2}-\frac{y_1+y_2}{2}\right)^2+\left(\frac{z_3}{2}-\frac{z_1+z_2}{2}\right)^2,$$

Whence,

$$\Sigma(\Delta^2)=\tfrac{1}{4}\{3\Sigma(x^2y^2z^2)-2\Sigma(x,y,z)\}.$$

Wherefore, finally,

$$\Sigma(\delta^2) = 4\Sigma(\Delta^2).$$

The solution sent by Mr. Reid was founded on the same general principle; and solutions for the case of the regular tetrahedron were given by Dr. Burns, and Messrs Harley and Wilkinson.

LXXXVII. *Mr. Thomas Weddle.*

An ellipse can be divided geometrically into n equal sectors when n is such, that a regular polygon of n sides can be described geometrically; required proof.

[Solution. *Mr. G. W. Hearn.*]

Let it be required to divide a given ellipse into n equal sectors by lines drawn from its centre, when n is such that a regular polygon of n sides can be described.

Let a and b be the semi-axes of the ellipse, and suppose a circle with radius b described whose plane is inclined at an angle $\cos^{-1}\frac{b}{a}$ to the plane of the ellipse, and so that the axis minor of the ellipse may be parallel to the plane of the circle, and the line joining the centres of the circle and ellipse at right angles to the plane of the former. Then the circle will be the orthographic projection of the ellipse, and any area in the plane of the ellipse projected on the plane of the circle will have to its projection the ratio $b : a$.

Hence if the ellipse be divided into equal sectors their projections will be equal sectors of the circle, and conversely, if a regular polygon be described

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in the circle, and lines drawn from its angular points perpendicular to the plane of circle, these lines will intersect the ellipse in points which being joined with its centre will divide it into equal sectors.

We may now conceive the circle and ellipse to be concentric, and a diameter of the circle consequently coincident with the axis minor of the ellipse, then, as in Descriptive Geometry, the sides of the orthographic cylinder will ultimately become straight lines parallel to the axis major of the ellipse.

Hence the following construction :—

On the axis minor of the ellipse describe a circle. Inscribe in it a regular polygon of n sides, and through its angular points draw lines parallel to the axis major of the ellipse, and join their points of intersection with the ellipse, and the centre, the ellipse will thus be divided into n equal sectors.

From similar considerations, if a circle be described on the major axis, and lines be drawn parallel to the minor axis through the angular points of the inscribed polygon, the same thing may be effected.

Also, since we may choose the first angle of the polygon inscribed in the circle arbitrarily, it follows not only that the ellipse may be geometrically divided into n equal sectors, but this in an infinite variety of ways.

Also that an ellipse may (in an infinite number of ways) be divided into sectoreal areas which shall have given ratios, provided a circle can be so divided.

MATHEMATICAL EXERCISES—*(continued.)*

93. *Mr. W. H. Levy, Shalbourne.*

Four times the sum of the squares of the distances of the centre of gravity from the angles of any plane quadrilateral, is equal to the sum of the squares of the four sides and the two diagonals, together with the square of the distance of the centre of gravity from the intersection of the diagonals.

94. *Mr. John Riddle, Nautical School, Greenwich.*

From the top of a mountain, h miles in height, the visible horizon appeared depressed $a°$; it is required to shew that, if d be the distance of the boundary of the visible horizon, and D the diameter of the earth,

$$d = h\cot\tfrac{1}{2}a \ldots\ldots (1) \qquad D + h = h\cot^2\tfrac{1}{2}a \ldots\ldots (2).$$

95. *Mr. R. H. Wright, London.*

An uniform beam whose length is a and weight W, rests with one end against a wall and is supported by a string from the other and passing over a pulley on the top of the wall with a weight P at its extremity; then if the part of the string that subtends the angle that the beam makes with the wall is l, the angle that the beam makes with the horizon ϕ, and h the perpendicular height of the pulley from the lower extremity of the beam when in equilibrium; shew that the inclination of the string to the beam will be

$$\tan^{-1}\left\{\frac{Wl}{2P(a\cos\phi \pm h\tan\phi)}\right\}.$$

96. *W. Burns, LL.D., St. Margaret's Bank, Rochester.*

Given the radius of a railway curve, the distance between the rails, the height of the outer above the level of the inner rail, and the height of the centre of gravity of the train, to find the velocity of the train when both rails sustain equal parts of the weight.

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97. *By Habitancum.*

Determine the values of h and k (these are the only variable quantities) that make the function,

$$u = (a-h)^2(\beta-k)^2-\{hk+\tfrac{1}{2}a(m+n)(a-2h)\}^2,$$

a maximum or minimum; u, a, β, m, n being *positive*, and h and k subject to the condition

$$mnu(a-2h)-\beta(\beta-2k)=0.$$

98. *Pen-and-Ink, Charlton.*

Two points and a straight line, not all in the same plane, being given, it is required to draw two lines from the given points to meet in the given line:

(1) That shall make equal angles with the line;

(2) That shall make the greatest angle with each other.

99. *Septimus Tebay.*

A circular plate, of equal density and thickness, is suspended horizontally by three equal vertical strings attached to three given points in its periphery. If it be twisted horizontally through a very small angle, so that its centre remains in the same vertical line, required the time of an oscillation.

100. *Mr. Weddle, Newcastle.*

Required the form of the equation to the hyperboloids whose asymptotic cones touch three given planes.

101. *Mr. Weddle.*

Show how an ellipsoid may be divided into n equal volumes.

102. *Mr. Fenwick.*

Find the least circumscribed ellipse to a given quadrilateral figure.

*** This problem has been solved by *Euler*, but it admits of a more concise and elegant investigation than the one given by that distinguished analyst.—*Nova Acta Acad. Petropol.* Vol. IX. p. 132.

103. *By J. J.*

In integrating the partial differential equation

$$\frac{dy}{dr}=x.\frac{dy}{dx}+y,$$

if x be considered a function of another variable v, and if $r+v=s$, and $r-v=s'$; prove that

$$\frac{d.xy}{d.s'}=0.$$

104. *Mr. G. W. Hearn, R. M. College.*

Find the moments of inertia of the five regular solids about any axis through their centres of gravity.

105. *By the same Gentleman.*

A cavity is scooped out of a solid sphere of given substance, and all that is known about it is, that it is one of the regular solids concentric with the sphere. It is required to determine by mechanical means (without opening or penetrating the sphere) the form and dimensions of the cavity.

106. *Mr. Davies.*

A complete spherical quadrilateral and its three diagonals are formed of great circles, and through the extremities of each diagonal and the intersections of the other two, other great circles are drawn: it is required to prove, independently of projective considerations, that all the circles concerned in the figure are so divided by each other, that the sines of their segments are in harmonical proportion.

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THE MATHEMATICIAN.

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ON THE GREATEST INSCRIBED, AND LEAST CIRCUMSCRIBED ELLIPSE, TO A GIVEN QUADRILATERAL FIGURE.

[*Mr. Fenwick.*]

The second of the following propositions was discussed by the celebrated EULER. Owing, however, to his peculiar mode of investigation, EULER'S solution is very long and complicated. See the *Nova Acta Acad. Petropol. vol.* IX., p. 132.

I am not aware that the problem, "To find the greatest inscribed ellipse to a given quadrilateral figure," has hitherto engaged the attention of any writer, though it leads to a discussion even more interesting than the other. Its investigation, it will be seen, results in a quadratic equation, one root of which gives the greatest inscribed ellipse, the other, an imaginary area. I have endeavoured to account for the origin of this imaginary expression.

1. *To find the greatest inscribed ellipse to a given quadrilateral figure*, PQRS.

Take lines through P, one of the angles of the quadrilateral, parallel to the sides SR, RQ, for axes of coordinates;

and let

$$x = a \ldots\ldots\ldots (1), \qquad y = \beta \ldots\ldots\ldots (2),$$
$$y = mx \ldots\ldots (3), \qquad y = nx \ldots\ldots (4)^*,$$

be the equations of the four sides QR, RS, PQ, PS.

Also, let (h, k) be the coordinates of the centre of an ellipse which is inscribed in PQRS; then this ellipse may be denoted by the equation

$$a(y-k)^2 + 2b(y-k)(x-h) + c(x-h)^2 + 1 = 0 \ldots\ldots\ldots (5),$$

where a, b, c, h, k are arbitrary constants.

We have also the relation

$$r^2 = (y-k)^2 + (x-h)^2 + 2(y-k)(x-h)\cos\theta \ldots\ldots\ldots (6),$$

θ being the angle contained by the axes, or the angle contained by the sides SR, RQ, and r any radius.

* Throughout this paper, m, n, α, β, are supposed to be *positive*, and hence the *salient* quadrilateral is referred to. The student will have no difficulty in adapting the resulting formulæ to the other cases.

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In order to express the *area* of the ellipse, we must find the product of the principal axes of (5). Moreover, the principal axes will be maxima or minima values of the radius, r. It follows, therefore, that the differential of (6) must be equated to zero; x, y, being subject to the equation of condition (5).

Differentiate (5) and (6) and we have the equations

$$a(y-k)\,dy + b(x-h)\,dy + b(y-k)\,dx + c(x-h)\,dx = 0 \ldots (7)$$

$$(y-k)\,dy + (x-h)\,dx + (y-k)\cos\theta\,dx + (x-h)\cos\theta\,dy = 0 \ldots (8)$$

Multiplying (8) by an arbitrary, λ, and adding the result to (7), gives, on equating to zero the coefficients of each differential,

$$\lambda(y-k) + \lambda(x-h)\cos\theta + a(y-k) + b(x-h) = 0 \ldots . (9)$$

$$\lambda(x-h) + \lambda(y-k)\cos\theta + b(y-k) + c(x-h) = 0 \ldots . (10)$$

Multiply (9) by $y-k$, (10) by $x-h$, and add the results, then by (5) and (6),

$$\lambda r^2 - 1 = 0, \text{ or } \lambda = \frac{1}{r^2}.$$

Substituting this value of λ in (9, 10), these equations become

$$\frac{1}{r^2}(y-k) + \frac{1}{r^2}(x-h)\cos\theta + a(y-k) + b(x-h) = 0,$$

$$\frac{1}{r^2}(x-h) + \frac{1}{r^2}(y-k)\cos\theta + b(y-k) + c(x-h) = 0;$$

from which we readily get on eliminating $x-h$, and dividing by $y-k$,

$$(ac-b^2)r^4 + (a+c-2b\cos\theta)\,r^2 + \sin^2\theta = 0 \ldots \ldots \ldots (11)$$

This is a quadratic equation in r^2, which will give the squares of the principal semi-axes, and hence by the theory of equations, the absolute term is equal to the product of the squares of the principal semi-axes of the ellipse. The area, therefore, of an inscribed ellipse is

$$\frac{\pi \sin\theta}{(ac-b^2)^{\frac{1}{2}}},$$

which is to be a maximum.

Hence, $ac-b^2$ must be a minimum ....(12)

We have now to determine the values of the quantities a, b, *etc.*

The conditions that the lines (1, 2, 3, 4) may be tangents to the curve (5), are

$$a\{c(\alpha-h)^2 + 1\} = b^2(\alpha-h)^2,$$

$$c\{a(\beta-k)^2 + 1\} = b^2(\beta-k)^2,$$

$$(am^2+2bm+c)(ak^2+2bhk+ch^2+1) = (amk+bmh+bk+ch)^2,$$

$$(an^2+2bn+c)(ak^2+2bhk+ch^2+1) = (ank+bnh+bk+ch)^2.$$

These equations readily reduce to the following:—

$$(b^2-ac)(\alpha-h)^2 = a \ldots \ldots \ldots \ldots \ldots \ldots (13)$$

$$(b^2-ac)(\beta-k)^2 = c \ldots \ldots \ldots \ldots \ldots \ldots (14)$$

$$(b^2-ac)(mh-k)^2 = am^2+2bm+c \ldots \ldots \ldots (15)$$

$$(b^2-ac)(nh-k)^2 = an^2+2bn+c \ldots \ldots \ldots \ldots (16)$$

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By (13) and (14)

$$(ac-b^2)^2 = \frac{ac}{(a-h)^2(\beta-k)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(17)$$

Hence by (12) and (17)

$$\frac{(ac)^{\frac{1}{2}}}{(a-h)\,(\beta-k)}, \text{ must be a minimum} \ldots\ldots\ldots(18)$$

Substitute the values of a and c from (13, 14) in (15, 16), transpose each term to the left which involves (b^2-ac); then dividing one resulting equation by the other, we get after reduction,

$$\beta(\beta-2k) = mna\,(a-2h) \ldots\ldots\ldots(19)$$

Again, equating the values of a deduced from (13, 15) and (13, 16), on eliminating b^2-ac, gives by means of (19)

$$\frac{c}{b} = -\frac{2(\beta-k)^2}{2hk+a(m+n)(a-2h)}.$$

Similarly, equations (14, 15) and (14, 16) give

$$\frac{a}{b} = -\frac{2(a-h)^2}{2hk+a(m+n)(a-2h)}.$$

Consequently,

$$\frac{ac}{b^2} = \frac{c}{b} \times \frac{a}{b} = \frac{4(a-h)^2(\beta-k)^2}{\{2hk+a(m+n)(a-2h)\}^2} \ldots\ldots\ldots\ldots(20)$$

Comparing now the values of b^2 in (17) and (20) we get

$$\frac{1}{ac-b^2} = \frac{(a-h)(\beta-k)}{(ac)^{\frac{1}{2}}}$$

$$=(a-h)^2(\beta-k)^2-\{hk+\tfrac{1}{2}a(m+n)(a-2h)\}^2,$$

which by (18) must be a maximum.

The differential of this expression, which we shall denote by u, taken with respect to h and k, on being reduced by (19), gives

$$\frac{du}{dh} = \frac{a}{2\beta^2}(ma-\beta)(\beta-na)\{3mn(a-2h)^2+2\beta(m+n)(a-2h)+\beta^2\};$$

and hence

$$\frac{d^2u}{dh^2} = -\frac{2a}{\beta^2}(ma-\beta)(\beta-na)\{3mn(a-2h)+\beta(m+n)\}.$$

The condition $\frac{du}{dh}=0$, gives

$$a-2h = -\frac{\beta}{3mn}\left\{m+n \pm \sqrt{(m^2+n^2-mn)}\right\},$$

which makes $\frac{d^2u}{dh^2} = \pm\frac{2a}{\beta^2}(ma-\beta)(\beta-na)\sqrt{(m^2+n^2-mn)}$.

Now ma, β and na, are obviously the ordinates parallel to the axis of y, of the points Q, R, and M the intersection of QR, and PS produced. It will therefore be evident, that β is *intermediate in value* to ma, and na; and

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consequently, the expression $(m\alpha-\beta)(\beta-n\alpha)$ is *positive*. And, moreover, since m^2+n^2-mn is *essentially positive*, the minus sign in the value of $\alpha-2h$ corresponds to a maximum, the positive one to a minimum.

The corresponding values of k are

$$\beta-2k=-\frac{a}{3}\{m+n\pm\sqrt{(m^2+n^2-mn)}\}.$$

It remains to be seen whether both these values of h and k make $ac-b^2$ *positive, the necessary condition for a real area*, as the expression for the area involves $(ac-b^2)^{\frac{1}{2}}$.

Employing the positive sign in the values of h and k, we have by (17)

$$\frac{1}{ac-b^2}=\frac{1}{108m^2n^2}\,a\beta\mu'(m\alpha-\beta)(\beta-n\alpha)\{6mn-(m+n)\mu'\}\ldots.(21),$$

where $$\mu'=m+n+\sqrt{(m^2+n^2-mn)}.$$

The negative sign gives

$$\frac{1}{ac-b^2}=\frac{1}{108m^2n^2}\,a\beta\mu(m\alpha-\beta)(\beta-n\alpha)\{6mn-(m+n)\mu\}\ldots.(22),$$

where $$\mu=m+n-\sqrt{(m^2+n^2-mn)}.$$

It is easy to see that (21) and (22) will be positive or negative according as the *last factor in each is positive or negative*. We shall therefore confine our inquiry to the sign of $6mn-(m+n)\mu'$, and $6mn-(m+n)\mu$.

Let $n-m=d$; then $n=m+d$, and $(m+n)^2=4mn+d^2$. Consequently, the last factor of the expression (21) becomes

$$2m(m+d)-\{d^2+(2m+d)\sqrt{(m^2+md+d^2)}\}\ldots\ldots\ldots(23)$$

Now, m^2+md+d^2 is greater than $m^2+md+\frac{1}{4}d^2$, or $(m+\frac{1}{2}d)^2$. Whence if

$$2m(m+d)-\{d^2+(2m+d)(m+\tfrac{1}{2}d)\}\ldots\ldots\ldots\ldots(24),$$

be negative, (23), and consequently (21), will also be negative.

But (24) is equal to $-\frac{1}{4}d^2$, a negative quantity. It follows, therefore, that (21) is *negative* and does not belong to the solution of the problem.

Again, the last factor of (22) is equivalent to

$$\begin{aligned}&6mn-(m+n)^2+(m+n)\sqrt{(m^2+n^2-mn)}\\&=2mn-(n-m)^2+(m+n)\sqrt{(m^2+n^2-mn)}\ldots\ldots(25)\end{aligned}$$

And since

m^2+n^2-mn is greater than m^2+n^2-2mn, or $(n-m)^2$,

and $2mn-(n-m)^2+(m+n)(n-m)=2m(2n-m)$, is *positive*,* (25) must also be positive. The expression (22) is therefore positive and hence fulfils the conditions.

Similar results are obtained whem m is greater than n.

Hence, *the greatest inscribed ellipse is the square root of*

$$\frac{\pi^2\sin^2\theta}{108m^2n^2}a\beta\mu(m\alpha-\beta)(\beta-n\alpha)\{6mn-(m+n)\mu\};$$

* We have tacitly assumed that n is greater than m by writing $n-m$ for the root of $(n-m)^2$, and hence $2m(2n-m)$ is positive.

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and *the coordinates of its centre are*

$$h = \frac{a}{2} + \frac{\beta\mu}{6mn}, \text{ and } k = \frac{\beta}{2} + \frac{a\mu}{6}.$$

The value of μ in these expressions is

$$\mu = m + n - \sqrt{(m^2 + n^2 - mn)}.$$

We may express the area, *etc.*, in terms of the sides and angles of the quadrilateral. For denoting the angles PQR, QRS, PSR, by Q, R, S, we have

$$\theta = \mathrm{R},\ a = \mathrm{PQ}\,\frac{\sin\mathrm{Q}}{\sin\mathrm{R}},\ \beta = \mathrm{PS}\,\frac{\sin\mathrm{S}}{\sin\mathrm{R}},$$

$$m = \frac{\sin(\mathrm{Q}+\mathrm{R})}{\sin\mathrm{Q}} \text{ and } n = \frac{\sin\mathrm{S}}{\sin(\mathrm{R}+\mathrm{S})}.$$

Scholium 1. The expression (21), as we have seen, does not fulfil the conditions of the problem, inasmuch as it gives an imaginary expression for the area of the inscribed ellipse. It evidently belongs to the hyperbola of which $ac - b^2$ is a maximum, and to which the sides of the given quadrilateral are tangential. We might indeed have anticipated such a result from the fundamental equation (5) which represents *central curves in general.*

We may now advert for a little to the line (19) which is the locus of the centres of all conic sections which touch the sides or the sides produced, of the given quadrilateral, PQRS.

The coordinates of the middle points of the diagonals PR, QS, are evidently

$$\frac{1}{2}a, \frac{1}{2}\beta;\ \text{and } \frac{1}{2}\left(a + \frac{\beta}{n}\right), \frac{1}{2}(ma + \beta);$$

and as these satisfy (19), it follows that

All conic sections that touch the sides or the sides produced of a given quadrilateral, have their centres situated on the straight line which bisects the diagonals of the quadrilateral.

This property has been noticed in respect of the *ellipse.* The hyperbolas obviously touch QR, RS; and PQ, PS produced.

2. *To find the least circumscribed ellipse to a given quadrilateral figure,* PQRS.

Take the diagonals PR, QS, which intersect in L, for axes of coordinates. Then all conic sections which pass through P,Q, R,S, may be denoted by the equation

$$ay^2 + 2bxy + cx^2 + 2dy + 2ex + 1 = 0 \quad \ldots\ldots\ldots\ldots (26),$$

where b is arbitrary, and the other coefficients are given by the equations

$$-\frac{e}{c} + \frac{1}{c}(e^2 - c)^{\frac{1}{2}} = \mathrm{LQ} = p \quad \ldots\ldots\ldots\ldots (27)$$

$$-\frac{e}{c} - \frac{1}{c}(e^2 - c)^{\frac{1}{2}} = \mathrm{LS} = q \quad \ldots\ldots\ldots\ldots (28)$$

$$-\frac{d}{a} + \frac{1}{a}(d^2 - a)^{\frac{1}{2}} = \mathrm{LR} = p' \quad \ldots\ldots\ldots\ldots (29)$$

$$-\frac{d}{a} - \frac{1}{a}(d^2 - a)^{\frac{1}{2}} = \mathrm{LP} = q' \quad \ldots\ldots\ldots\ldots (30)$$

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If now (m,n) be the coordinates of the centre of (26), that equation will assume the form

$$a(y-n)^2+2b(y-n)(x-m)+c(x-m)^2+f=0 \ldots\ldots\ldots (31),$$

of which

$$f=\frac{2bed+ac-ae^2-cd^2-b^2}{ac-b^2}.$$

We have also the relation (r being any radius, and ϕ the angle contained by the diagonals),

$$r^2=(y-n)^2+(x-m)^2+2(y-n)(x-m)\cos\phi \ldots\ldots\ldots (32)$$

Operating with (31,32) as with (5,6) of last article, we get the equation

$$(ac-b^2)r^4+f(a+c-2b\cos\phi)r^2+f^2\sin^2\phi=0;$$

the last term of which is obviously equal to the product of the squares of the principal semi-axes of the circumscribed ellipse.

It follows, therefore, that

$$\frac{\pi f\sin\phi}{(ac-b^2)^{\frac{1}{2}}},$$

must be a minimum.

The differential of this expression taken with respect to b, being equated to zero, gives the following cubic equation for determining b:—

$$b^3-4deb^2+\{3(ae^2+cd^2)-ac\}b-2acde=0 \ldots\ldots\ldots (33)$$

Let $p=\frac{1}{2}$, $q=-\frac{1}{3}$, $p'=1$, $q'=-\frac{1}{5}$; then (27, 28, 29, 30)

$$a=-5,\ c=-6,\ d=2,\ e=\frac{1}{2}.$$

The equation (33) now becomes

$$4b^3-16b^2-423b-240=0.$$

We find by the ordinary methods that one positive root of this equation is between 12 and 13, a negative root between 8 and 9, and another negative root between ·5 and ·6; the last of which only fulfils the conditions, as the other two make $ac-b^2$ *negative*.

These values of p, q, p', q', constitute the *salient* quadrilateral. The *re-entrant* one is evidently excluded, the equation of the conic section in such case being of the form

$$ay^2+2bxy-cx^2+2dy+2ex+1=0,$$

which does not denote an ellipse whatever be the value of b.

Again, let $p=\frac{1}{2}$, $q=\frac{1}{3}$, $p'=1$, $q'=\frac{1}{5}$; then

$$a=5,\ c=6,\ d=-3,\ e=-\frac{5}{2}.$$

The equation (33) now reduces to

$$4b^3-120b^2+903b-1800=0.$$

The roots of this equation are all real, but one only fulfils the conditions. The second set of values of p, q, *etc.*, that we have employed, constitute the *intersectant* quadrilateral.

It would seem then, that analytically considered there are three minimum circumscribed ellipses to a given quadrilateral figure; one only of these,

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however, satisfies the conditions of the problem. The other results most probably arise from the general nature of the equation (31), which represents *hyperbolas* as well as ellipses.

Scholium 2. EULER as I have stated has given a long solution of this problem (his solution occupies about six quarto pages), in the *ninth volume of the Petersburgh Transactions*, who, after giving in general the resulting equation, thus comments on it:—

"Ecce ergo tota solutio problematis propositi perducta est ad resolutionem æquationis cubicæ, quae cum semper habeat radicem realem, certum est, quomodocunque quatuor puncta fuerint disposita, semper unam ellipsin assignari posse per quatuor illa puncta transeuntem, cujus area omnium sit minima, pro qua æquatio inter coordinatas x et y exhiberi poterit, si modo loco b radix ex illa æquatione cubica oriunda substituatur. Quodsi forte eveniat ut æquatio illa cubica tres admittat radices reales, totidem quoque solutiones locum habebunt, quarum autem indolem aliis perscrutandam relinquo."

R. M. Academy, October 16th, 1846.

MATHEMATICAL NOTES.

I. *To find the relations between the oblique conjugate diameters and principal diameters of the ellipsoid.*

Referring the ellipsoid to any three conjugate diameters as oblique axes, its equation is of the form

$$\frac{z^2}{c^2} + \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Then r being any radius, and f, g, h, the cosines of the angles between the axes (y, z), (x, z), (x, y), respectively, we have the relation

$$r^2 = z^2 + y^2 + x^2 + 2xyh + 2xzg + 2yzf \quad \ldots\ldots\ldots\ldots (2)$$

It will be obvious that the *principal axes* will be maxima or minima values of the radius, r, and hence, the differential of (2) must be equated to zero; x, y, z being subject to the equation of condition (1).

Differentiate (1) and (2), and we have the following equations,

$$\frac{z}{c^2}\,dz + \frac{y}{b^2}\,dy + \frac{x}{a^2}\,dx = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$z\,dz + y\,dy + x\,dx + h\,x\,dy + h\,y\,dx + g\,x\,dz + g\,z\,dx + f\,y\,dz + f\,z\,dy = 0 \ldots (4)$$

Multiplying (4) by an arbitrary λ, and adding the result to (3), gives, on equating to zero the co-efficients of each differential,

$$\lambda(z + gx + fy) + \frac{z}{c^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$\lambda(y + hx + fz) + \frac{y}{b^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$\lambda(x + hy + gz) + \frac{x}{a^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

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Multiply (5) by z, (6) by y, (7) by x, and add the results; then by (1) and (2),

$$\lambda r^2 + 1 = 0, \quad \text{or } \lambda = -\frac{1}{r^2}.$$

Substituting this value of λ in (5, 6, 7), these equations become

$$\left(1 - \frac{r^2}{c^2}\right)z + fy + gx = 0,$$
$$fz + \left(1 - \frac{r^2}{b^2}\right)y + hx = 0,$$
$$gz + hy + \left(1 - \frac{r^2}{a^2}\right)x = 0.$$

Eliminating x, y, z from these equations, and arranging for r, we get the equation

$$r^6 - (a^2+b^2+c^2)r^4 - \{a^2b^2(1-h^2)+a^2c^2(1-g^2)+b^2c^2(1-f^2)\}r^2$$
$$-a^2b^2c^2(1-f^2-g^2-h^2+2fgh) = 0 \ldots\ldots\ldots\ldots (8)$$

The absolute term of this equation is obviously equal to the product of the squares of the principal semi-axes of the ellipsoid.

Hence if A, B, C, be the principal semi-axes of (1), one of the relations in question is

$$ABC = abc(1-f^2-h^2-g^2+2fgh)^{\frac{1}{2}}.$$

For another investigation of the relations between the conjugate and principal diameters of the ellipsoid, I beg to refer the inquiring student to *Gregory's Solid Geometry*, page 88.

II. The following proof of two known properties of the ellipse, may be interesting to some of our readers.

Referring the ellipse to two conjugate diameters as oblique axes, its equation is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Also, r being any radius and θ the angle contained by the axes,

$$r^2 = x^2 + y^2 + 2xy\cos\theta \ldots\ldots\ldots\ldots\ldots (2)$$

Differentiating (1) and (2), and proceeding as in the preceding note, we get the equation

$$r^4 - (a^2+b^2)r^2 + a^2b^2\sin^2\theta = 0 \ldots\ldots\ldots\ldots\ldots (3)$$

Hence, if r_1, r_2, be the principal semi-axes of (1), we have by the theory of equations,

$$r_1^2 r_2^2 = a^2b^2\sin^2\theta,$$
or $$r_1 r_2 = ab\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Also, $$-r_1^2 + r_1r_2 - r_1r_2 - r_1r_2 + r_1r_2 - r_2^2 = -(a^2+b^2),$$
or $$r_1^2 + r_2^2 = a^2 + b^2 \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The properties alluded to above are contained in (4) and (5).

S. F.

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APPLICATION OF ALGEBRA TO THE MODERN GEOMETRY.

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(Continued from p. 147.)

Relations of the segments into which an arbitrary transversal is cut by a system of surfaces of the n^{th} degree.

XXVIII.

If a straight line AP be cut by a given surface of the n^{th} degree in the points $Q_1, Q_2, Q_3 .. Q_n$, it is required to express the ratio

$$PQ_1.PQ_2.PQ_3..PQ_n : AQ_1.AQ_2.AQ_3..AQ_n \text{ or } (PQ_n):(AQ_n),^*$$

which is compounded of the ratios of the segments of AP, in terms of the coordinates of P referred to any rectangular axes passing through A.

Let Ax, Ay, Az be the coordinate axes, and put $AP = r$, $AQ_1 = \rho_1$, $AQ_2 = \rho_2$, $AQ_3 = \rho_3 .. AQ_n = \rho_n$; then

$$\frac{(PQ_n)}{(AQ_n)} = \frac{(r-\rho_1)(r-\rho_2)(r-\rho_3)..(r-\rho_n)}{\rho_1.\rho_2.\rho_3..\rho_n};$$

and when n is even this reduces to

$$\frac{(PQ_n)}{(AQ_n)} = 1-\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}+etc.\right)r+\left(\frac{1}{\rho_1\rho_2}+\frac{1}{\rho_1\rho_3}+\frac{1}{\rho_2\rho_3}+etc.\right)r^2- etc.\ldots\ldots (a).$$

Let $u = 0$ be the equation of the given surface; then

$$u = 1 + f_1(x, y, z)+f_2(x, y, z)+f_3(x, y, z)..+f_n(x, y, z)=0..(b),$$

where $f_m(x, y, z)$ denotes a complete homogeneous function of x, y, z of the m^{th} degree, such as

$$Ax^m + By^m + Cz^m + A'x^{m-1}y + A''x^{m-1}z + B'y^{m-1}x + etc.:$$

but if ρ denote the radius vector of the surface, and λ, μ, ν the cosines of the angles which it makes with the axes Ax, Ay, Az, we shall have

$$x = \lambda\rho,\ y = \mu\rho,\ z = \nu\rho;$$

and by substituting these values of x, y, z in (b) we obtain

$$1+\rho f_1(\lambda, \mu, \nu)+\rho^2 f_2(\lambda, \mu, \nu)+\rho^3 f_3(\lambda, \mu, \nu)\ldots.+\rho^n f_n(\lambda, \mu, \nu) = 0,$$

which is the polar equation of the given surface. Now since $\rho_1, \rho_2 \ldots. \rho_n$ are the roots of this equation, we obtain, by the theory of equations,

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}+etc. = -f_1(\lambda, \mu, \nu),$$

$$\frac{1}{\rho_1\rho_2}+\frac{1}{\rho_1\rho_3}+\frac{1}{\rho_2\rho_3}+etc. = f_2(\lambda, \mu, \nu,), \text{ etc.};$$

hence, by substituting these expressions in (a), we get

$$(PQ_n):(AQ_n)= 1 + rf_1(\lambda, \mu, \nu)+r^2 f_2(\lambda, \mu, \nu)\ldots.+r^n f_n(\lambda, \mu, \nu);$$

* The symbol (PQ_n) is here used to denote the continued product of n segments of a straight line having the first letter P for a common extremity, and the second letter Q with the numbers $1, 2, 3 \ldots. n$ suffixed for their other extremities. Thus $(PQ_3) = PQ_1.PQ_2.PQ_3$, $(P_4Q_4) = P_4Q_1.P_4Q_2.P_4Q_3.P_4Q_4$, *etc.* This notation is already in common use, and as it greatly simplifies our formulas, and abridges the labour of the printer, it will be frequently used in the remaining part of this section.

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and if $x'y'z'$ be the rectangular coordinates of P, so that $\lambda r = x'$, $\mu r = y'$, $\nu r = z'$, this equation may be written in the form

$$(PQ_n):(AQ_n) = 1 + f_1(x',y',z') + f_2(x',y',z') \ldots + f_n(x',y',z') = u';$$

where u' denotes the same function of $x'y'z'$ as u is of xyz.

When n is odd, it may be demonstrated in the same manner that $(PQ_n):(AQ_n) = -u'$; hence, generally, we have

$$\frac{PQ_1.PQ_2.PQ_3 \ldots PQ_n}{AQ_1.AQ_2.AQ_3 \ldots AQ_n} = \pm u' \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where the upper or lower sign must be taken according as n is even or odd.

XXIX.

Let $u = 0$ be an equation of the n^{th} degree in x,y,z, $v = 0$ one of the m^{th} degree, where m is not greater than n, and p a constant quantity; then the equation

$$u = pv \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

being satisfied by $u = 0$, $v = 0$, evidently represents a surface of the n^{th} degree passing through the intersection of the surfaces $u = 0$ and $v = 0$.

To determine the relations between the points in which the three surfaces are cut by an arbitrary transversal, let a straight line be drawn from the common origin A, meeting the surface $u=0$ in the points $Q_1, Q_2, Q_3 \ldots Q_n$, the surface $v=0$ in $R_1, R_2, R_3 \ldots R_m$, and the surface (2) in $P_1, P_2 \ldots P_n$; then, by equation (1), $\pm u = (P_1Q_n):(AQ_n)$, $\pm v = (P_1R_m):(AR_m)$, and by substituting these values of u and v in (2), we get

$$(P_1Q_n)(AR_m) = \pm p(P_1R_m)(AQ_n);$$

similarly $$(P_2Q_n)(AR_m) = \pm p(P_2R_m)(AQ_n),$$

and by eliminating p we obtain

$$(P_1Q_n)(P_2R\ ) = (P_2Q_n)(P_1R_m) \ldots\ldots\ldots\ldots (3);$$

consequently the points Q and R together with P_1 and P_2 form a system of $m+n+2$ points in involution.

Hence *if a surface of the n^{th} degree pass through the intersection of two given surfaces not exceeding the n^{th} degree, and if any transversal be drawn meeting the three surfaces, the points in which it cuts any two of the surfaces and any two of the points in which it cuts the third surface, are in involution.*

Since the points P, Q, R, taken two at a time, give $\frac{1}{2}n(n-1)$, $\frac{1}{2}n(n-1)$, $\frac{1}{2}m(m-1)$ combinations respectively; the transversal contains $n(n-1)$ systems of $m+n+2$ points in involution, and $\frac{1}{2}m(m-1)$ systems of $2n+2$ points in involution.

This proposition is analogous to the theorem of STURM relative to a system of three conic sections passing through the same four points. We shall see hereafter that it is true when one or both of the surfaces are complex.

XXX.

If the equations $u = 0$, $v = 0$ denote any two surfaces of the n^{th} degree, the equations

$$(a) \ldots\ldots u = pv, \qquad u = p'v \ldots\ldots\ldots (b)$$

will evidently represent two surfaces of the n^{th} degree, each of which passes through the intersection of the surfaces $u = 0$ and $v = 0$. Let any straight line, drawn from the common origin A, meet the surface $u = 0$ in the points

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$P_1, P_2, P_3 \ldots P_n$, the surface $v = 0$ in $Q_1, Q_2, Q_3 \ldots Q_n$, the surface (a) in $R_1, R_2 \ldots R_n$, and the surface (b) in $S_1, S_2 \ldots S_n$; then from (a) and (b), we have

$$\frac{(R_1P_n)}{(AP_n)} = p\frac{(R_1Q_n)}{(AQ_n)}, \quad \frac{(S_1P_n)}{(AP_n)} = p'\frac{(S_1Q_n)}{(AQ_n)};$$

hence, by dividing the former by the latter, we obtain

$$\frac{(R_1P_n)}{(R_1Q_n)} : \frac{(S_1P_n)}{(S_1Q_n)} = \frac{p}{p'} \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

When the four surfaces $u = 0$, $v = 0$, (a) and (b) are planes, this becomes

$$\frac{R_1P_1}{R_1Q_1} : \frac{S_1P_1}{S_1Q_1} = \frac{p}{p'} \ldots\ldots\ldots\ldots\ldots\ldots (4'),$$

and the first member of this equation has been called by CHASLES the anharmonic relation of the four points P_1, Q_1, R_1, S_1. Hence the first member of (4) may be called *the anharmonic function of* $2n + 2$ points, and the property implied in that equation may be expressed as follows;—

If any four surfaces of the n^{th} *degree pass through a curve of double curvature of the degree* n^2, *and if a transversal meet any two of the surfaces in* $2n$ *points and each of the other surfaces in one point, the anharmonic function of these points is invariable, whatever be the direction of the transversal.*

The number of combinations obtained by varying R and S will evidently be n^2, and since the points R and S may be interchanged with P and Q, and with Q and R, it follows that the anharmonic function of $2n+2$ points may be expressed in $3n^2$ different ways. When the surfaces are planes, $n = 1$ and the anharmonic ratio may be expressed in three different ways.—(*Mathematician*, Vol. I, p. 171.)

When $p' = -p$, equation (4′) shows that R_1 and S_1 are conjugate harmonic points with respect to P_1 and Q_1; and equation (4) becomes

$$(R_1P_n)(S_1Q_n) + (R_1Q_n)(S_1P_n) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

From the analogy of this equation to the particular case which has just been noticed, its first member may be called the *harmonic function* of $n + 2$ points; and the surfaces $u = 0$, $v = 0$, $u - pv = 0$, $u + pv = 0$, may be considered as a system of *harmonic radial surfaces* of the n^{th} degree.

If R_1 and S_1 be replaced by R_2 and S_2, we obtain from (4)

$$\frac{(R_2P_n)}{(R_2Q_n)} : \frac{(S_2P_n)}{(S_2Q_n)} = \frac{p}{p'};$$

and by comparing this with (4) we get

$$(R_1P_n)(S_2P_n)(R_2Q_n)(S_1Q_n) = (R_2P_n)(S_1P_n)(R_1Q_n)(S_2Q_n) \ldots (6),$$

which is a relation of involution of $2n + 4$ points.

XXXI.

If the equations $u = 0$, $v = 0$, $w = 0$ denote three surfaces, the first of which is of the n^{th}, and the other two of the m^{th} degree, ($2m$ being not greater than n), the equation

$$u = pvw \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

(where p is a given number) will denote a surface of the n^{th} degree passing through the curves $u=0$, $v=0$, and $u=0$, $w=0$, in which the surface $u=0$

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is cut by the surfaces $v = 0$ and $w = 0$. Now if the last two surfaces coincide, the two curves of intersection will coalesce in a curve of contact; and therefore the equation

$$u = pv^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

denotes a surface of the n^{th} degree having a contact of the first order with the surface $u = 0$, along the curve of intersection of the surfaces $u = 0$ and $v = 0$. Again, if $v' = 0$ denote a surface of the r^{th} degree, where r is not greater than m, the equation

$$u = p'v'^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7')$$

will evidently represent a surface of the n^{th} degree touching the surface $u = 0$ along the curve of intersection of the surfaces $u = 0$ and $v' = 0$. Now by subtracting (7') from (7) we get $pv^2 - p'v'^2 = 0$, which may be resolved into the two equations

$$v\sqrt{p} + v'\sqrt{p'} = 0 \text{ and } v\sqrt{p} - v'\sqrt{p'} = 0 \ldots\ldots\ldots\ldots (8);$$

and since these equations denote two surfaces of the m^{th} degree, passing through the curve of intersection of the surfaces $v = 0$ and $v' = 0$, we have the following theorem (XXX):—

If each of two surfaces of the n^{th} degree touch a given surface of the n^{th} degree along the curves in which it is cut by two other surfaces, one of which is of the m^{th} degree and the other of the r^{th} degree, ($2m$ being not greater than n, and r not greater than m); the enveloping surfaces intersect upon two surfaces of the m^{th} degree, which pass through the intersection of the surfaces of contact, and form with them a system of four harmonic radial surfaces.

This is analogous to the system of MONGE (VI).

Let a transversal be drawn from the origin A to meet the surface $u = 0$ in $Q_1, Q_2 \ldots Q_n$, the surface $v = 0$ in $R_1, R_2 \ldots R_m$, and the surface (7) in $P_1, P_2 \ldots P_n$; then, in virtue of (1), equation (7) gives

$$(P_1Q_n):(AQ_n) = \pm p(PR^2_m):(AR^2_m), \quad (P_2Q_n):(AQ_n) = \pm p(P_2R^2_m):(AR^2_m);$$

$$\therefore \frac{P_1Q_1.P_1Q_2.P_1Q_3 \ldots P_1Q_n}{P_1R_1^2.P_1R_2^2.P_1R_3^2 \ldots P_1R^2_m} = \frac{P_2Q_1.P_2Q_2.P_2Q_3 \ldots P_2Q_n}{P_2R_1^2.P_2R_2^2.P_2R_3^2 \ldots P_2R^2_m} :$$

consequently the points P_1, P_2 together with the points Q and R form a system of $n+m+2$ points in involution; and each of the points R is double.

XXXII.

To find the equation of a surface of the n^{th} degree, passing through m given plane sections of a given surface of the n^{th} degree, when m is not greater than n.

Let $u = 0$ be the equation of the given surface, $v_1 = 0$, $v_2 = 0$, $v_3 = 0 \ldots$ $v_m = 0$ those of the given planes, and ϕ a function of x, y, z not exceeding the $(n-m)^{\text{th}}$ degree; then the equation of the required surface will be

$$u = p.v_1.v_2.v_3. \ldots v_m.\phi \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

p being a given quantity. For this equation is evidently satisfied by the following systems of simultaneous equations;—

$$\left.\begin{matrix} u = 0 \\ v_1 = 0 \end{matrix}\right\}, \quad \left.\begin{matrix} u = 0 \\ v_2 = 0 \end{matrix}\right\}, \quad \left.\begin{matrix} u = 0 \\ v_3 = 0 \end{matrix}\right\} \ldots\ldots \left.\begin{matrix} u = 0 \\ v_n = 0 \end{matrix}\right\} \ldots\ldots (a),$$

and therefore the surface (9) passes through the m plane curves represented by these equations.

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Since equation (9) is also satisfied by the simultaneous equations $u = 0$ and $\phi=0$, it is evident that the surface (9) passes through the intersection of these two surfaces; the latter of which cannot exceed the $(n-m)^{\text{th}}$ degree, as otherwise (9) would exceed the n^{th} degree. Hence *if a surface of the n^{th} degree pass through m plane curves of the n^{th} degree which lie upon a given surface of the n^{th} degree, the other curve in which it intersects that surface lies upon a surface of the $(n-m)^{\text{th}}$ degree.*

When $m = n - 1$ the function ϕ is of the first degree with respect to x, y, z, and therefore represents a plane. Hence *if a surface of the n^{th} degree intersect a given surface of the n^{th} degree in $n-1$ plane curves, its other intersection with the surface is also a plane curve.*

In this case, if we put $u = t.v_1.v_2.v_3.\ldots.v_{n-1}$, equation (9) becomes $t = p\phi$, and since ϕ is of the first degree, this equation may be written in the form

$$t = ax + by + cz + d \ldots\ldots\ldots\ldots\ldots (9').$$

Now if the surface (9′) pass through two given points $x'y'z'$ and $x''y''z''$, we have the conditions

$$ax'+by'+cz'+d=t', \quad ax''+by''+cz''+d=t'';$$

from which we infer as in No. 11, that the plane $\phi = 0$ passes through a fixed point which ranges on the straight line joining the two given points. Hence *if a surface of the n^{th} degree pass through two given points, and intersect a given surface of the n^{th} degree in $n-1$ plane curves, the remaining plane in which it intersects the given surface will pass through a fixed point in the straight line joining the two given points.*

If a straight line be drawn from the origin A, meeting the surface $u = 0$ in $Q_1, Q_2, Q_3 \ldots Q_n$, the system of planes in $R_1, R_2, R_3 \ldots R_m$, and the surface $\phi=0$ in $S_1, S_2, S_3 \ldots S_{n-m}$; it is evident from (1) that equation (9) may be written in the form

$$(P_1Q_n)(AR_m)(AS_{n-m})=p\,(P_1R_m)(P_1S_{n-m})(AQ_n), \text{ or}$$
$$(P_2Q_n)(AR_m)(AS_{n-m})=p\,(P_2R_m)(P_2S_{n-m})(AQ_n);$$

hence by eliminating p we obtain

$$(P_1Q_n)(P_2R_m)(P_2S_{n-m}) = (P_2Q_n)\,(P_1R_m)(P_1S_{n-m}) \ldots\ldots\ldots (10),$$

and therefore the $2n$ points Q,R,S, together with P_1 and P_2 are in involution.

XXXIII.

When the planes $v_2, v_3, v_4 \ldots v_m$ coincide with v_1, and ϕ is constant, equation (9) becomes $u=pv_1^m$; and the curves of intersection (a) unite in a curve of contact of the $(m-1)^{\text{th}}$ order; hence the equation

$$u = v^m \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

represents a surface of the n^{th} degree, having a contact of the $(m-1)^{\text{th}}$ order with the surface $u=0$ along the curve in which that surface is cut by the plane $v=0$. Similarly the equation

$$u = v'^m \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11')$$

represents a surface touching the surface $u = 0$ along the curve in which it is cut by the plane $v' = 0$. Now by subtracting the latter equation from

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the former we obtain $v^m - v'^m = 0$, which may be resolved into the two equations

$$v - v' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

and $v^{m-1} + v^{m-2}v' + v^{m-3}v'^2 + v^{m-4}v'^3 \ldots v'^{m-1} = 0 \ldots\ldots\ldots (b)$;

and, when m is even, the latter of these may be resolved into

$$v + v' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c),$$

$$v^{m-2} + v^{m-4}v'^2 + v^{m-6}v'^4 \ldots v'^{m-2} = 0 \ldots\ldots\ldots\ldots (d);$$

hence, *if two surfaces of the n^{th} degree have a plane contact with a given surface of the n^{th} degree, their intersections lie upon two planes, or upon one plane, passing through the intersection of the planes of contact, according as the contact is of an odd or an even order.*

If $t^m = u$, equation (11) gives $t = v$, or

$$t = ax + by + cz + d:$$

hence if the surface (11) pass through two given points $x'y'z'$ and $x''y''z''$, we have the conditions

$$ax' + by' + cz' + d = t', \quad ax'' + by'' + cz'' + d = t'';$$

from which we infer, as in No. VII., that the plane $v = 0$ passes through a fixed point in the straight line joining the two given points. Hence *if a surface of the n^{th} degree pass through two given points and touch a given surface of the n^{th} degree along a plane curve, without cutting the surface in any other curve, the plane of contact will pass through a fixed point which ranges on the straight line joining the two given points.*

XXXIV.

If $u = 0$ be an equation of the n^{th} degree and $v = 0$ one of the $(n-1)^{th}$ degree, the equation

$$u = pv \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

will evidently denote a surface of the n^{th} degree passing through the intersection of the surfaces $u = 0$, $v = 0$, and also cutting the surface $u = 0$ in a plane curve of the n^{th} degree at infinity. Let a transversal drawn from A in any direction meet the surface (12) in $P_1, P_2 \ldots P_n$, the surface $u = 0$ in $Q_1, Q_2 \ldots Q_n$, the surface $v = 0$ in $R_1, R_2 \ldots R_{n-1}$, and the plane at infinity in R_n; then since the points P, Q with R_1 and R_n are in involution (XXXII), we shall have

$$(R_1P_n) : (R_1Q_n) = (R_nP_n) : (R_nQ_n) = 1, \text{ (since } R_n \text{ is at infinity)};$$

$$\therefore (R_1P_n) = (R_1Q_n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

and consequently R_1 is a centre of involution of the $2n$ points P and Q. Hence *if the intersection of a surface of the n^{th} degree with another surface of the n^{th} degree lie on a surface of the $(n-1)^{th}$ degree only, and if any transversal be drawn cutting the three surfaces, each of the points in which it cuts the surface of the $(n-1)^{th}$ degree is a centre of involution of the points in which it cuts the other two surfaces: and therefore that surface may be considered as the locus of centres of involution for the two surfaces of the n^{th} degree.* This proposition holds when the surface $v = 0$ is complex; as, for instance, when it is composed of a system of $n-1$ planes.

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XXXV.

To find the equation of a surface of the n^{th} degree passing through mn of the intersections of $m+n$ given planes.

Let the equations of the given planes be

$$v_1=0,\ v_2=0,\ v_3=0\ldots\ldots v_{m+n}=0;$$

then the equation of the required surface will be

$$v_1.v_2.v_3\ldots.v_n=p.v_{n+1}v_{n+2}v_{n+3}\ldots.v_{n+m}\phi\ldots\ldots\ldots\ldots(14),$$

where p is constant and ϕ a function of x, y, z not exceeding the $(n-m)^{th}$ degree. For, since equation (14) is satisfied by the following systems of simultaneous equations

$$\left.\begin{matrix}v_1=0\\v_{n+1}=0\end{matrix}\right\}\quad\left.\begin{matrix}v_1=0\\v_{n+2}=0\end{matrix}\right\}\quad\left.\begin{matrix}v_1=0\\v_{n+3}=0\end{matrix}\right\}\cdots\cdots\cdots\left.\begin{matrix}v_1=0\\v_{n+m}=0\end{matrix}\right\}$$

$$\left.\begin{matrix}v_2=0\\v_{n+1}=0\end{matrix}\right\}\quad\left.\begin{matrix}v_2=0\\v_{n+2}=0\end{matrix}\right\}\quad\left.\begin{matrix}v_2=0\\v_{n+3}=0\end{matrix}\right\}\cdots\cdots\cdots\left.\begin{matrix}v_2=0\\v_{n+m}=0\end{matrix}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left.\begin{matrix}v_n=0\\v_{n+1}=0\end{matrix}\right\}\quad\left.\begin{matrix}v_n=0\\v_{n+2}=0\end{matrix}\right\}\quad\left.\begin{matrix}v_n=0\\v_{n+3}=0\end{matrix}\right\}\cdots\cdots\cdots\left.\begin{matrix}v_n=0\\v_{n+m}=0\end{matrix}\right\}$$

it follows that the surface (14) passes through the $m\,n$ straight lines represented by these equations.

Since $m+n$ quantities taken n at a time give $\dfrac{(m+n)(m+n-1)\ldots(m+1)}{1.2\ldots.n}$ combinations, it is evident that the same number of systems of surfaces of the n^{th} degree may be described such that every surface in each of the systems shall pass through mn of the intersections of $m+n$ given planes. When $m=n$ these systems recur, and the number of distinct systems is $\dfrac{1}{2}\dfrac{(n+1)(n+2)..2n}{1.2.3..n}$.

When $m=n-1$, the function ϕ cannot exceed the first degree. Hence *if a surface of the n^{th} degree pass through $n\,(n-1)$ of the intersections of $2n$ given planes, it will in general pass through n other intersections of these planes.*

Let a straight line be drawn from the origin A, meeting the planes $v_1=0$, $v_2=0..v_n=0$ in $Q_1,Q_2..Q_n$, the planes $v_{n+1}=0$, $v_{n+2}=0..v_{n+m}=0$ in $R_1, R_2..R_m$, the surface $\phi=0$ in $S_1, S_2..S_{n-m}$, and the surface (14) in $P_1, P_2..P_n$; then, in virtue of (1), equation (14) gives

$$(P_1Q_n)(AR_m)(AS_{n-m})=p.(AQ_n)(P_1R_m)(P_1S_{n-m}),$$

$$(P_2Q_n)(AR_m)(AS_{n-m})=p.(AQ_n)(P_2R_m)(P_2S_{n-m}),$$

$$\therefore\quad (P_1Q_n)(P_2R_m)(P_2S_{n-m})=(P_2Q_n)(P_1R_m)(P_1S_{n-m})\ldots\ldots(15),$$

and consequently the n points Q, the m points R, the $n-m$ points S, and the two points P_1 and P_2 are in involution.

When the planes $v_{n+1}=0$, $v_{n+2}=0..v_{n+m}=0$ coincide, the mn straight lines, through which the surface (14) passes, unite in n straight lines of contact, and equation (14) becomes

$$v_1.v_2.v_3.\ldots.v_n=p(v_{n+1})^m\phi\ldots\ldots\ldots\ldots\ldots\ldots(16);$$

which therefore represents a surface of the n^{th} degree touching the planes $v_1=0$, $v_2=0,..v_n=0$, along the lines in which they are cut by the plane

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$v_n + 1 = 0$. In this case the points $R_1, R_2, \ldots R_m$ coincide, and the relation of involution becomes

$$P_2R_1^m(P_1Q_n)(P_2S_{n-m}) = P_1R_1^m(P_2Q_n)(P_1S_{n-m}) \ldots\ldots\ldots (15'),$$

R_1 being a multiple point.

This proposition is analogous to the problem *ad quatuor lineas* of PAPPUS, as also to the properties of rule surfaces of the second degree given in the first and second part of this section.

XXXVI.

Let the equations

$$u_1 = 0,\ u_2 = 0,\ u_3 = 0 \ldots\ldots\ldots u_n = 0,$$

referred to the same rectangular axes, denote a system of surfaces of the 1^{st}, 2^{nd}, 3^{rd}, $\ldots\ldots n^{th}$ degree, respectively; and let a straight line AP_1 be drawn from the origin A, meeting the surface $u_1 = 0$ in Q_1, $u_2 = 0$ in R_1, R_2, $u_3 = 0$ in S_1, S_2, S_3, and so on; then if the point P_1 be taken so that

$$1 + \frac{P_1Q_1}{AQ_1} + \frac{P_1R_1.P_1R_2}{AR_1.AR_2} + \frac{P_1S_1.P_1S_2.P_1S_3}{AS_1.AS_2.AS_3} + etc. = 0 \ldots . (17)$$

the equation of the locus of P_1 will be (XXVIII)

$$1 - u_1 + u_2 - u_3 \ldots\ldots \pm u_n = 0 \ldots\ldots\ldots\ldots\ldots (18)$$

and therefore the locus of P_1 is a surface of the n^{th} degree.

Let us first consider the case of two surfaces $u_m = 0$, $u_n = 0$, m being less than n. In this case we have

$$1 + (P_1T_m):(AT_m) + (P_1V_n):(AV_n) = 0 \ldots\ldots\ldots (17'),$$

$$1 \pm u_m \pm u_n = 0 \ldots\ldots\ldots\ldots\ldots (18').$$

Let $P_2, P_3, \ldots P_n$ be the points in which the other surface (18') is cut by AP_1, then if p and q denote the reciprocals of (AT_m) and (AV_n) we obtain from (17')

$$\left.\begin{aligned} 1 + p(P_1T_m) + q(P_1V_n) &= 0 \\ 1 + p(P_2T_m) + q(P_2V_n) &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ 1 + p(P_nT_m) + q(P_nV_m) &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (a).$$

By eliminating p and q from the first three of these equations we get

$$(P_1T_m)\{(P_2V_n) - (P_3V_n)\} + (P_2T_m)\{P_3V_n) - (P_1V_n)\} + (P_3T_m)\{P_1V_n) - (P_2V_n)\} = 0 \ldots . (19);$$

and since the n equations (a) taken three at a time give $\frac{n(n-1)(n-2)}{1.2.3}$ different combinations, we shall evidently have an equal number of equations similar to (19). These equations express a property of algebraic surfaces, which belongs to the same class as the relations of involution, and appears to approach nearest to those relations in point of simplicity.

If we take the equation of four terms

$$1 \pm u_l \pm u_m \pm u_n = 0 \ldots\ldots\ldots\ldots\ldots (18'')$$

equation (17) gives, in this case,

$$\left.\begin{aligned} 1 + p(P_1R_l) + q(P_1T_m) + r(P_1V_n) &= 0 \\ 1 + p(P_2R_l) + q(P_2T_m) + r(P_2V_n) &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ 1 + p(P_nR_l) + q(P_nT_m) + r(P_nV_n) &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (b),$$

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hence by eliminating p,q,r from these equations, taken four at a time, we shall obtain $\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$ equations analogous to (19), but much more complicated owing to the greater number of their terms.

It is evident that when any of the terms of (17) or (18) are wanting it is always possible to eliminate the fixed point A, and obtain one or more relations involving only the segments of the transversal made by the given system of surfaces.

Equation (1) enables us to extend this investigation so as to determine the projective properties of the segments of a transversal cut by a system of surfaces

$$u_1 = 0,\ u_2 = 0,\ u_3 = 0, \ldots\ldots u_n = 0,\ v = 0,$$

when v is any algebraic function of $u_1, u_2, u_3, \ldots\ldots u_n$. The surface $v=0$ may in this case be considered as a central projection of $u_n = 0$, made by means of the directing surfaces $u_1 = 0$, $u_2 = 0$, *etc.*; and therefore equation (1) seems to afford a much more general theory of central projection than any that has hitherto been noticed. We shall see hereafter that PONCELET'S *figures homologiques* are only a very simple class of the surfaces defined by the equation $v = f\ (u_1, u_2, u_3, etc.)$, where f is any algebraic function.

XXXVII.

Before closing this section it may be proper to show how equation (1) may be generalized so as to comprehend the general theory of transversals. For this purpose I shall now proceed to solve the following problem;—

If a straight line AP be cut by a surface of the n^{th} degree in $Q_1, Q_2, Q_3 \ldots Q_n$, it is required to express the ratio compounded of the ratios of its segments in terms of the coordinates of A and P referred to any rectangular axes Ox, Oy, Oz.

Let $AP = r$, $AQ_1 = \rho_1$, $AQ_2 = \rho_2, \ldots\ldots AQ_n = \rho_n$; then, when n is even, we have

$$\frac{(PQ_n)}{(AQ_n)} = 1 - \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} + etc.\right)r + \left(\frac{1}{\rho_1\rho_2} + \frac{1}{\rho_1\rho_3} + \frac{1}{\rho_2\rho_3} + etc.\right)r^2 - etc. \quad \ldots\ldots(a).$$

Let the equation of the given surface be $u = f(x, y, z) = 0$, and let ρ be the radius vector of the surface measured from A, and λ, μ, ν the cosines of the angles which it makes with Ox, Oy, Oz, so that if $x_1\ y_1\ z_1$ be the coordinates of A, we shall have

$$x - x_1 = \lambda\rho,\ y - y_1 = \mu\rho,\ z - z_1 = \nu\rho;$$

then if $u_1 = f(x_1, y_1, z_1)$, we obtain by TAYLOR'S theorem

$$u = u_1 + \rho\left(\frac{du_1}{dx_1}\lambda + \frac{du_1}{dy_1}\mu + \frac{du_1}{dz_1}\nu\right)$$
$$+ \frac{1}{2}\rho^2\left(\frac{d^2u_1}{dx_1^2}\lambda^2 + \frac{d^2u_1}{dy_1^2}\mu^2 + \frac{d^2u_1}{dz_1^2}\nu^2 + 2\frac{d^2u_1}{dx_1dy_1}\lambda\mu + 2\frac{d^2u_1}{dx_1dz_1}\lambda\nu + 2\frac{d^2u_1}{dy_1dz_1}\mu\nu\right)$$
$$+ \frac{1}{1.2.3}\rho^3\left(\frac{d^3u_1}{dx_1^3}\lambda^3 + \frac{d^3u_1}{dy_1^3}\mu^3 + \frac{d^3u_1}{dz_1^3}\nu^3 + etc.\right) + etc. = 0,$$

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which is the polar equation of the given surface. Now since ρ_1, ρ_2, *etc.* are the roots of this equation, we obtain by the theory of equations

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}+\frac{1}{\rho_3}+etc. = -\frac{1}{u_1}\left(\frac{du_1}{dx_1}\lambda+\frac{du_1}{dy_1}\mu+\frac{du_1}{dz_1}\nu\right),$$

$$\frac{1}{\rho_1\rho_2}+\frac{1}{\rho_1\rho_3}+\frac{1}{\rho_2\rho_3}+etc. = \frac{1}{1.2.u_1}\left(\frac{d^2u_1}{dx_1^2}\lambda^2+\frac{d^2u_1}{dy_1^2}\mu^2+\frac{d^2u_1}{dz_1^2}\nu^2+etc.\right), etc.$$

and by substituting these expressions in (a) we get

$$\frac{(PQ_n)}{(AQ_n)} = 1+\frac{r}{u_1}\left(\frac{du_1}{dx_1}\lambda+\frac{du_1}{dy_1}\mu+\frac{du_1}{dz_1}\nu\right)$$
$$+\frac{r^2}{1.2.u_1}\left(\frac{d^2u_1}{dx_1^2}\lambda^2+\frac{d^2u_1}{dy_1^2}\mu^2+\frac{d^2u_1}{dz_1^2}\nu^2+etc.\right)+etc.$$

But since r is the radius vector of P measured from A, we have

$$\lambda r = x'-x_1,\ \mu r = y'-y_1,\ \nu r = z'-z_1,$$

where $x'y'z'$ are the coordinates of P, and therefore

$$u_1(PQ_n):(AQ_n)=u_1+\frac{du_1}{dx_1}(x'-x_1)+\frac{du_1}{dy_1}(y'-y_1)+\frac{du_1}{dz_1}(z'-z_1)$$
$$+\tfrac{1}{2}\frac{d^2u_1}{dx_1^2}(x'-x_1)^2+etc. \ldots\ldots(a').$$

Let $u'=f(x',y',z')=f(x_1+x'-x_1,\ y_1+y'-y_1,\ z_1+z'-z_1)$, then by TAYLOR'S theorem

$$u'=u_1+\frac{du_1}{dx_1}(x'-x_1)+\frac{du_1}{dy_1}(y'-y_1)+\frac{du_1}{dz_1}(z'-z_1)+\tfrac{1}{2}\frac{d^2u_1}{dx_1^2}(x'-x_1)^2+etc.;$$

hence the second member of (a') is u', and we have $(PQ_n):(AQ_n)=u':u_1$. In like manner we should find, when n is odd, $(PQ_n):(AQ_n)=-u':u_1$, and therefore, generally, we have

$$\frac{PQ_1.PQ_2.PQ_3..PQ_n}{AQ_1.AQ_2.AQ_3..AQ_n}=(-1)^n\frac{u'}{u_1}\ldots\ldots\ldots\ldots(20).$$

The following examples will be sufficient to show that the most general properties of transversals may be deduced from this equation as simple corollaries.

XXXVIII.

Let $u=f(x,y,z)=0$ be the equation of a surface of the n^{th} degree; A, B, C, D, E any points in space, and let AB meet the surface in the points $P_1,P_2..P_n$, BC in $Q_1,Q_2..Q_n$, CD in $R_1,R_2..R_n$, and DE in $S_1, S_2..S_n$; then if $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$, $x_4y_4z_4$, $x_5y_5z_5$ be the coordinates of A,B,C,D,E, respectively, and if for brevity we put $u_m=f(x_m, y_m, z_m)$, we shall have (20)

$$\frac{(BP_n)}{(AP_n)}=\frac{u_2}{u_1},\ \frac{(CQ_n)}{(BQ_n)}=\frac{u_3}{u_2},\ \frac{(DR_n)}{(CR_n)}=\frac{u_4}{u_3},\ \frac{(ES_n)}{(DS_n)}=\frac{u_5}{u_4};$$

hence, by multiplying these equations we obtain

$$\frac{(BP_n)(CQ_n)(DR_n)(ES_n)}{(AP_n)(BQ_n)(CR_n)(DS_n)}=\frac{u_5}{u_1}\ldots\ldots\ldots\ldots\ldots(21).$$

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Since the second member of this equation is independent of the coordinates of B, C, D, if the points A and E be fixed while B, C and D change their position, it is evident that the function in the first member of the equation will be invariable.

When E coincides with A, or the polygon is closed, equation (21) becomes

$$(BP_n)(CQ_n)(DR_n)(AS_n) = (AP_n)(BQ_n)(CR_n)(DS_n).....(22);$$

hence, if the sides of *any* polygon be cut by a transversal surface of the n^{th} degree, the continued product of the conterminous segments measured from the angular points of the polygon taken in order, remains the same when that order is reversed.

When the transversal surface is a plane we have

$$BP_1.CQ_1.DR_1.AS_1 = AP_1.BQ_1.CR_1.DS_1.....(23),$$

hence if the sides of any polygon be cut by a transversal plane, the continued product of all the segments which have no common extremity is equal to the continued product of all the rest.

The last two theorems are due to CARNOT.

XXXIX.

Equation (20) is applicable to transcendental curves and surfaces as well as algebraic ones. Thus let the proposed curve be the quadratrix of TSCHIRNHAUSEN represented by the equation

$$y = a\cos\frac{\pi x}{2a}.....(a).$$

Let b and c be the coordinates of any point B in the first positive quadrant, and through B draw AB parallel to the axis of x, meeting the axis of y in A, and the curve in P_1, P_2, P_3, *etc.* in the positive direction, and in Q_1, Q_2, Q_3, *etc.* in the negative direction; then by equation (20)

$$\frac{BP_1.BQ_1.BP_2.BQ_2.BP_3.BQ_3.\ etc.}{AP_1.AQ_1.AP_2.AQ_2.AP_3.AQ_3.\ etc.} = \frac{c-a\cos\frac{\pi b}{2a}}{c-a}.....(24).$$

To reduce this to algebraic form, we observe that when $y = c$ equation (a) gives

$$x = \pm\frac{2a}{\pi}\left(\cos^{-1}\frac{c}{a} \pm 2m\pi\right),$$

where m may be any whole positive number. Hence, putting $\cos^{-1}\frac{c}{a} = \theta$, we shall have

$$AP_1 = 2a\theta:\pi \quad = -AQ_1,$$
$$AP_2 = 4a-2a\theta:\pi = -AQ_2,$$
$$AP_3 = 4a+2a\theta:\pi = -AQ_3,$$
$$AP_4 = 8a-2a\theta:\pi = -AQ_4,\ etc.;$$

but $\frac{BP_1}{AP_1} = 1-\frac{b}{AP_1}$, $\frac{BQ_1}{AQ_1} = 1+\frac{b}{AP}$, $\frac{BP_2}{AP_2} = 1-\frac{b}{AP_2}$, $\frac{BQ_2}{AQ_2} = 1+\frac{b}{AP_2}$, *etc.*

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and by substituting these expressions in (24) we get

$$\frac{c-a\cos\frac{\pi b}{2a}}{c-a}=\left(1-\frac{b\pi}{2a\theta}\right)\left(1+\frac{b\pi}{2a\theta}\right)\left(1-\frac{b\pi}{4a\pi-2a\theta}\right)\left(1+\frac{b\pi}{4a\pi-2a\theta}\right)$$
$$\times\left(1-\frac{b\pi}{4a\pi+2a\theta}\right)\left(1+\frac{b\pi}{4a\pi+2a\theta}\right)\left(1-\frac{b\pi}{8a\pi-2a\theta}\right),\ etc..(25)$$

When $c=0$ we have $\theta=\frac{1}{2}\pi$; hence, in this case, (25) becomes

$$\cos\frac{\pi b}{2a}=\left(1-\frac{b}{a}\right)\left(1+\frac{b}{a}\right)\left(1-\frac{b}{3a}\right)\left(1+\frac{b}{3a}\right)\left(1-\frac{b}{5a}\right),\ etc.$$

and as this formula is well known it may be considered as a verification of the more general equations (24) and (25). As all the known methods of demonstrating this last equation are in some degree objectionable, the method of demonstration here given may perhaps be preferred by some readers.

When the line AB is not parallel to the axis of x, an infinite number of the points P and Q are in general imaginary; and it is extremely difficult to verify (24), owing to the difficulty of finding the roots (whether real or imaginary) of transcendental equations.

XL.

Let $u=0$ denote a surface of the n^{th} degree, $v=0$ one of the m^{th} degree, and $w=0$ one of the r^{th} degree; and from any three fixed points A, B, C draw AM meeting the surface $u=0$ in the points $P_1, P_2, \ldots P_n$, BM meeting $v=0$ in $Q_1, Q_2, \ldots Q_m$, and CM meeting $w=0$ in $R_1, R_2, \ldots R_r$; then if $x_1y_1z_1$ be the coordinates of A, $x_2y_2z_2$ those of B, and $x_3y_3z_3$ those of C, we shall have (XXXVII)

$$\frac{(MP_n)}{(AP_n)}=\pm\frac{u'}{u_1},\ \frac{(MQ_m)}{(AQ_m)}=\pm\frac{v'}{v_1},\ \frac{(MR_r)}{(AR_r)}=\pm\frac{w'}{w_1};$$

hence if the ratio compounded of the first two ratios be supposed to be a given multiple of the duplicate of the third ratio, the equation to the locus of M will be

$$u'v'=pw'^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26),$$

where p is a given number; therefore the locus touches the surfaces $u'=0$ and $v'=0$ along the curves in which they are cut by the surface $w'=0$.

This example may perhaps be sufficient to illustrate the method of applying equation (20) to any particular case of the following general problem:—

To find the locus of the point M such that if straight lines be drawn from it to any number of given points each cutting a given surface in as many points (real or imaginary) as there are units in its degree, the continued product of the segments of one secant which have a common extremity at M may be a given function of the similar continued products of the segments of the other secants.

(To be continued.)

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NOTE ON PRATT'S DEMONSTRATION OF THE PARALLELOGRAM OF FORCES.

[*Mr. J. F. Heather, A.M., Royal Military Academy.*]

Pratt in his "Mechanical Philosophy" attempts to give a completely analytical proof of the principle of the parallelogram of forces, and founds it upon an assumption, which he expresses in the following terms:— "That a force can produce no effect in a direction perpendicular to its own direction." Now I think I shall be able to show that this expression must either have no meaning at all, or that it must be taken in the same sense as though we should say, "a force can produce no effect in any direction whatever except its own." For if it be meant to say that a force cannot be resolved into two others, the direction of one of which shall be at right angles to its own direction, the assertion would be manifestly incorrect; but, if it be only meant to say that a number of forces, having their directions all lying along two straight lines at right angles to each other, must have the algebraic sum of the forces acting along the one of these lines, and also the sum of those acting along the other separately equal to zero, in order that the whole may be in equilibrium, then the proposition is equally true of a number of forces having their directions all lying along any two straight lines, inclined at any angle whatever. This is evident from the equation $R^2 = P^2 + Q^2 + 2PQ\cos\alpha$: and also follows immediately, from the consideration, that any two forces acting upon a point, in directions not along the same straight lines must have a finite resultant; for if the forces acting along any two lines are not separately in equilibrium, they all resolve into two forces, which not acting in opposite directions, along the same straight line, cannot counteract each other's effects. The values of $f(\theta)$ for the particular values of θ set down by Mr. Pratt, are nevertheless true, because in each of these particular cases, one of the two directions into which the power P is resolved, is in the same line with its own direction, in which case the component along the other direction vanishes, whatever be the angle which this direction makes with the direction of P.

PRACTICAL METHOD OF FORMING LOGARITHMS AND ANTI-LOGARITHMS, INDEPENDENTLY OF EXTENSIVE TABLES.

[*Professor Hearn, Royal Military College.*]

The accompanying concise tables are intended for obtaining the logarithms of numbers consisting of ten or less than ten digits to ten places of the mantissa of the logarithm. If fewer places be required the tables are of course equally available, and the indispensable calculation considerably curtailed. Again, if logs. to 20 figures were required, tables for finding them might easily be framed on the same principle, and they would be but double the extent of those here presented.

In the "Anti-Logarithmic" set, Table I. contains the logarithms of the nine digits; II. contains the logarithms of 11, 12 *etc.* to 19 inclusive; III. contains those of 101, 102, 103 *etc.* to 109 inclusive; IV. contains those of 1001, 1002, 1003 *etc.* to 1009 inclusive, and so on.

In the "Logarithmic" set, Table I. contains the arithmetical complements of the logarithms of $10-1$, $10-2$, $10-3$ *etc.* to $10-9$, or of 9, 8, 7, *etc.* to 1; II. contains the ar. comp. log. of $100-1$, $100-2$ *etc.* to $100-9$, or 99, 98, 97 *etc.* to 91; III. those of $1000-1$, $1000-2$ *etc.* to $1000-9$, or 999, 998 *etc.* to 991, and so on.

After this brief but sufficient description we will take at random ten

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figures, find their logarithms, and thence conversely the corresponding numbers.

Required the log. of 3475986427 to ten places.

3475986427		
2433190499	I. 7	.5228787453
1042795928		
417 11837	II. 4	177287670
1001084091		
1001084	III. 1	04345118
1000083007		
80006	V. 8	347449
1000003001		
	VI. 3	13029
	IX. 1	4
∴ log. 3475986427	=	9·5410780723

In this the proposed number is first multiplied by such a digit that when the product is written underneath it one place further to the right and subtracted, the first figure of the remainder may be 1 and the next 0, or as small as possible. It is found that 7 is the requisite multiplier so that the first remainder is the proposed number multiplied by 1 — ·7 or ·3. Table I. of the "Logarithmic" set is then referred to and the log. opposite 7 taken out. The first remainder is then multiplied by such a figure that when placed two figures to the right the second remainder shall have 100 for initial figures. This number is found to be 4. Table II. is referred to and the log. opposite 4 taken out. The process is continued on the second remainder, *etc.*

Conversely, let it be required to find the number whose log. is 9·5410780723. The index being 9 the number consists of ten digits.

	·5410780723
I. 3	639568176
II. 1	225641324
III. 5	13748333
IV. 3	739003
V. 1	304730
VI. 7	735
VIII. 1	301
IX. 6	40
X. 9	1

The number required is therefore to be ascertained from the product

$3 \times 11 \times 105 \times 1003 \times 10001 \times 100007 \times 10000001 \times 100000006 \times 1000000009$

$3 \times 11 \times 105 = 3465$

	3465...
3	10395
	3475395...
1	347539
	3475742539
7	243302
	3475985841
1	348
	3475986189
6	208
	3475986397
9	30
	3475986427

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The superfluous labour of writing again the figures already settled may be avoided thus:—

	34	65
3		10 395
	75	395
1		34 7 539
		74 2 539
7		24 3 302
		98 5 841
1		348
		6 189
6		208
		397
9		30
		427

In this, after rejecting the index, we refer to the first table of the "Anti-Logarithmic" set and note the logarithm next less than the given one. Then put down the difference; then note the log. in II. next less than this difference and subtract it, putting down the second difference, and so on. By this means we determine a series of factors which may be multiplied with the greatest ease, and thus the number is determined.

The following exhibits the work necessary when six places only are wanted:—

Required the log. of 48·76 to six places.

	4876..		
I. 7	34132	I. 7	522879
	14628.		
I. 3	43884	I. 3	154902
	102396		
II. 2	2048	II. 2	08774
	100348		
	301	III. 3	1305
	100047		
		IV. 4	174
		V. 7	30
	∴ log. 48·76	=	1·688064

Conversely, required the number whose log. is 1·688064.

I. 4	688064	4 × 12 × 101 × 1005 × 10007 × 100007 × 1000004	
II. 2	86004		
	4848..		
III. 1	6823	2424	
	2502	7224	
IV. 5	336	341	
	32	565	∴ 4·87601 is required number.
V. 7	2	34	
VI. 7		99	
VII. 4		2	
		601	

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LOGARITHMIC TABLES.

I. 10^1—

1	04575	74906
2	09691	00130
3	15490	19600
4	22184	87496
5	30102	99957
6	39794	00087
7	52287	87453
8	69897	00043
9		

II. 10^2—

1	0436	48054
2	0877	39243
3	1322	82657
4	1772	87670
5	2227	63947
6	2687	21464
7	3151	70514
8	3621	21727
9	4095	86077

III. 10^3—

1	043	45118
2	086	94587
3	130	48417
4	174	06616
5	217	69193
6	261	36156
7	305	07515
8	348	83278
9	392	63455

IV. 10^4—

1	04	34316
2	08	68676
3	13	03029
4	17	37525
5	21	72015
6	26	06549
7	30	11126
8	34	75746
9	39	10410

V. 10^5—

1	0	43430
2	0	86860
3	1	30290
4	1	73721
5	2	17152
6	2	60585
7	3	04017
8	3	47449
9	3	90883

VI. 10^6—

1	04343	
2	08686	
3	13029	
4	17372	
5	21715	
6	26058	
7	30401	
8	34744	
9	39087	

VII. 10^7—

1	0434	
2	0869	
3	1303	
4	1737	
5	2171	
6	2606	
7	3040	
8	3474	
9	3909	

VIII. 10^8—

1	043	
2	087	
3	130	
4	174	
5	217	
6	261	
7	304	
8	347	
9	391	

IX. 10^9—

1	04	
2	09	
3	13	
4	17	
5	22	
6	26	
7	30	
8	35	
9	39	

X. 10^{10}—

1	0	
2	1	
3	1	
4	2	
5	2	
6	3	
7	3	
8	4	
9	4	

ANTI-LOGARITHMIC TABLES.

I. 0

1		
2	30102	99957
3	47712	12547
4	60205	99913
5	69897	00043
6	77815	12504
7	84509	80400
8	90308	99870
9	95424	25094

II. 10^1

1	04139	26852
2	07918	12460
3	11394	33523
4	14612	80357
5	17609	12591
6	20411	99827
7	23044	89214
8	25527	25051
9	27875	36010

III. 10^2

1	0432	13738
2	0860	01718
3	1283	72247
4	1703	33393
5	2118	92991
6	2530	58653
7	2938	37777
8	3342	37555
9	3742	64979

IV. 10^3

1	043	40775
2	086	77215
3	130	09330
4	173	37128
5	216	60618
6	259	79807
7	302	94706
8	346	05321
9	389	11662

V. 10^4

1	04	34273
2	08	68502
3	13	02688
4	17	36830
5	21	70930
6	26	04986
7	30	38998
8	34	72967
9	39	06895

VI. 10^5

1	0	43429
2	0	86858
3	1	30286
4	1	73714
5	2	17142
6	2	60569
7	3	03995
8	3	47422
9	3	90847

VII. 10^6

1	04343	
2	08686	
3	13029	
4	17372	
5	21715	
6	26058	
7	30400	
8	34743	
9	39086	

VIII. 10^7

1	0434	
2	0868	
3	1303	
4	1737	
5	2171	
6	2606	
7	3040	
8	3474	
9	3909	

IX. 10^8

1	043	
2	087	
3	130	
4	174	
5	217	
6	261	
7	304	
8	347	
9	391	

X. 10^9

1	04	
2	09	
3	13	
4	17	
5	22	
6	26	
7	30	
8	35	
9	39	

XI. 10^{10}

1	0	
2	1	
3	1	
4	2	
5	2	
6	3	
7	3	
8	4	
9	4	

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ON THE PECULIAR EQUATION $x^3 - bx = 1$.

[*James Lockhart, Esq.*]

1. Let $p^3 - rp = s$ be a cubic equation having three real roots, and let it be represented by

$$p^3 - \left(\frac{b^4 - 6b}{3}\right)p = \frac{2b^6}{27} - \frac{2b^3}{3} + 1 \ldots\ldots\ldots\ldots (A),$$

then will

$$\frac{r^3}{s^2} = 6{\cdot}75 + \frac{b^3 - 6{\cdot}75}{s^2} \ldots\ldots\ldots\ldots (B),$$

where b is consequently greater than $(6{\cdot}75)^{\frac{1}{3}}$. If $b = 9$, the quotient is 6·750000...., and the greater b is assumed, the number of *zeros* following 6·75 increases in a known ratio, for such number is obvious when the values of b and s are given. The number of *zeros* would be unlimited if the magnitude of b were unlimited, because s is greater than b.

2. The equation $x^3 - bx = 1$ is brought into relation with $p^3 - rp = s$, by assuming

$$x = \frac{p + \frac{b^2}{3}}{1 - bp - \frac{b^3}{3}} \ldots\ldots\ldots\ldots\ldots\ldots (C).$$

Let $x^3 - 30x = 1$; then $b = 30$, and by (A) we get the equation

$$p^3 - 269940p = 53982001;$$

and by (B) the quotient

$$\frac{r^3}{s^2} = 6{\cdot}75000000000\ldots\ldots$$

When the roots of the equation $x^3 - bx = c$ are equal, then $\frac{b^3}{c^2} = 6{\cdot}75$ exactly, and the positive root $= \frac{3c}{b}$.

The equation in p is so closely allied to this case that $\frac{3s}{r}$ can be safely taken as the positive root in p to a great extent of figures. The result is 599·933329, which is so far correct, and had the division been further extended, it would probably have afforded a still closer approximation.

Applying this value of p to the formula (C), it gives $x = -{\cdot}033334568038$, which is one of the roots of the equation $x^3 - 30x = 1$, and correct to the last decimal figure.

When one of the roots is known, the others are derived by applying such root to the formula $-\frac{x}{2} \pm \left\{b - \frac{3x^2}{4}\right\}^{\frac{1}{2}}$. The remaining roots in p are $-299{\cdot}9664599$ and $-299{\cdot}966869$, which, if somewhat more extended, would give the other roots in x.

3. The equation $x^3 + bx = 1$ may be treated in the same manner as the foregoing equation, for the three formulas remain unaltered, but b and its odd powers are here negative.

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Let, for example, $x^3 + 12x = 1$ be the proposed equation; then

$$p^3 - 6936p = 222337;$$
$$\therefore\ p = 96{\cdot}1665224, \text{ and thence } x = {\cdot}083285191....$$

There are some very remarkable features connected with the resolution of all cubic equations, arising from the present case; but these are reserved for another communication.

THE COORDINATES OF THE CENTRE OF GRAVITY OF A FIGURE BOUNDED BY STRAIGHT LINES OR PLANES.

[*Professor Finlay.*]

1. When a body is composed of any number of parts, the centres of gravity of which are known, the coordinates of its centre of gravity are easily determined. Let w be the weight of one of these parts, xyz the coordinates of its centre of gravity; w' the weight of another part, $x'y'z'$ the coordinates of its centre of gravity; and so on; and let $x_1y_1z_1$ be the coordinates of the centre of gravity of the whole body: then, by the theory of parallel forces,

$$\left.\begin{aligned} x_1 &= \frac{wx + w'x' + w''x'' + etc.}{w + w' + w'' + etc.} \\ y_1 &= \frac{wy + w'y' + w''y'' + etc.}{w + w' + w'' + etc.} \\ z_1 &= \frac{wz + w'z' + w''z'' + etc.}{w + w' + w'' + etc.} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (A);$$

and when w, w', w'', *etc.*, are all equal, we have

$$\left.\begin{aligned} x_1 &= \frac{x + x' + x'' + etc.}{n} \\ y_1 &= \frac{y + y' + y'' + etc.}{n} \\ z_1 &= \frac{z + z' + z'' + etc.}{n} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (B),$$

where n is the number of parts of which the body is composed.

2. This being premised, let it be required to find the coordinates of the centre of gravity of a uniform physical straight line. The centre of gravity of a straight line evidently coincides with the centre of gravity of two equal weights placed at its extremities; hence the coordinates of the centre of gravity of the line will be obtained by taking $n = 2$ in equations (B), and we have

$$x_1 = \frac{x + x'}{2},\quad y_1 = \frac{y + y'}{2},\quad z_1 = \frac{z + z'}{2} \quad \ldots\ldots\ldots\ldots (1),$$

where $x\,y\,z$, $x'y'z'$ are the coordinates of the extremities of the line.

3. To find the centre of gravity of the perimeter of any rectilineal figure.

Let xyz, $x'y'z'$, *etc.*, be the coordinates of the angular points of the polygon; then, by equations (1), $\frac{1}{2}(x+x')$, $\frac{1}{2}(y+y')$, $\frac{1}{2}(z+z')$ are the coordinates of the centre of gravity of the first side; $\frac{1}{2}(x'+x'')$, $\frac{1}{2}(y'+y'')$, $\frac{1}{2}(z'+z'')$ those of the centre of gravity of the second side, and so on.

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Hence if $w, w', etc.$ be the weights of the 1st, 2nd, *etc.* sides, and $x_1y_1z_1$ the coordinates of the centre of gravity of the perimeter of the polygon, we obtain from $(\mathcal{A})$

$$\left.\begin{aligned} x_1 &= \frac{w(x+x')+w'(x'+x'')+w''(x''+x''')+etc.}{2(w+w'+w''+etc.)} \\ y_1 &= \frac{w(y+y')+w'(y'+y'')+w''(y''+y''')+etc.}{2(w+w'+w''+etc.)} \\ z_1 &= \frac{w(z+z')+w'(z'+z'')+w''(z''+z''')+etc.}{2(w+w'+w''+etc.)} \end{aligned}\right\} \ldots\ldots (2)$$

When all the sides of the polygon lie in the same plane, the axes of x and y may be supposed to lie in that plane; and the first two of these equations determine the centre of gravity.

4. To find the coordinates of the centre of gravity of a triangular plate of uniform thickness and density; the thickness of the plate being supposed to be indefinitely small.

By a well known theorem the centre of gravity of the triangular plate coincides with the centre of gravity of three equal weights placed at its angular points; and therefore the coordinates of the centre of gravity of the triangle will be obtained from (B), by taking $n = 3$. Thus, if xyz, $x'y'z'$, $x''y''z''$ denote the angular points of the triangle, and $x_1y_1z_1$ its centre of gravity, we shall have

$$x_1 = \frac{x+x'+x''}{3}, \quad y_1 = \frac{y+y'+y''}{3}, \quad z_1 = \frac{z+z'+z''}{3} \quad \ldots\ldots (3).$$

5. To find the coordinates of the centre of gravity of the surface of any rectilineal figure.

Let $xyz, x'y'z', etc.$ be the coordinates of the angular points of the polygon; then if the figure be resolved into triangles by straight lines drawn from the point xyz to all the remote angular points, the coordinates of the centre of gravity of the first triangle will be (4)

$$\tfrac{1}{3}(x+x'+x''), \quad \tfrac{1}{3}(y+y'+y''), \quad \tfrac{1}{3}(z+z'+z'');$$

those of the centre of gravity of the second triangle will be

$$\tfrac{1}{3}(x+x''+x'''), \quad \tfrac{1}{3}(y+y''+y'''), \quad \tfrac{1}{3}(z+z''+z''') \; etc.$$

Hence if $w, w', w'', etc.$ denote the weights of the triangles, and $x_1y_1z_1$ the centre of gravity of the polygon, we obtain from $(\mathcal{A})$

$$\left.\begin{aligned} x_1 &= \frac{w(x+x'+x'')+w'(x+x''+x''')+w''(x+x'''+x'''')+etc.}{3(w+w'+w''+etc.)} \\ y_1 &= \frac{w(y+y'+y'')+w'(y+y''+y''')+w''(y+y'''+y'''')+etc.}{3(w+w'+w''+etc.)} \\ z_1 &= \frac{w(z+z'+z'')+w'(z+z''+z''')+w''(z+z'''+z'''')+etc.}{3(w+w'+w''+etc.)} \end{aligned}\right\} \ldots (4)$$

When the sides of the polygon are in the same plane, its centre of gravity may be found from the first two of these equations, by taking the plane of the polygon as the plane of xy.

6. To find the coordinates of the centre of gravity of a homogeneous triangular pyramid.

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It is well known that the centre of gravity of a triangular pyramid coincides with the centre of gravity of four equal weights placed at the angular points of the pyramid; therefore, by taking $n = 4$ in (B) we obtain the following expressions for the coordinates of the centre of gravity of the pyramid;—

$$x_1 = \frac{x+x'+x''+x'''}{4},\ y_1 = \frac{y+y'+y''+y'''}{4},\ z_1 = \frac{z+z'+z''+z'''}{4} \ldots (5),$$

where xyz, $x'y'z'$, $x''y''z''$, $x'''y'''z'''$ denote the angular points of the pyramid.

7. To find the coordinates of the centre of gravity of any polyhedron.

Let xyz, $x'y'z'$, *etc.*, represent the angular points of the polyhedron; then if the figure be resolved into triangular pyramids by planes drawn from the point xyz through the remote angular points, the coordinates of the centre of gravity of each of the pyramids so formed become known from equations (5). Hence if w, w', *etc.*, be the weights of the pyramids, and $x_1y_1z_1$ the centre of gravity of the polyhedron, we shall have (A)

$$\left.\begin{aligned} x_1 &= \frac{w(x+x'+x''+x''')+w'(x+x''+x'''+x'''')+etc.}{4(w+w'+etc.)} \\ y_1 &= \frac{w(y+y'+y''+y''')+w'(y+y''+y'''+y'''')+etc.}{4(w+w'+etc.)} \\ z_1 &= \frac{w(z+z'+z''+z''')+w'(z+z''+z'''+z'''')+etc.}{4(w+w'+etc.)} \end{aligned}\right\} \ldots . (6).$$

8. *Scholium.*—Since each of the triangular pyramids of which the polyhedron is composed is supposed to be homogeneous, the weights w, w', *etc.* are proportional to the volumes of the pyramids; and these may readily be expressed in terms of the coordinates $x\,y\,z$, $x'y'z'$, *etc.* Hence equations (6) determine the coordinates of the centre of gravity of the polyhedron as functions of the coordinates of its angular points. It is evident also that (2) and (4) determine the coordinates of the centre of gravity of the perimeter and surface of a polygon, as functions of the coordinates of its angular points; since the quantities w, w', *etc.*, are proportional to the lengths of the sides of the polygon in the former case; and to the areas of the triangles of which it is composed in the latter; and both of these are easily expressed in terms of those coordinates.

By comparing equation (B) with (4) it will readily appear that the centre of gravity of a system of equal weights placed at the angular points of a polygon is, in general, different from the centre of gravity of its surface. From a remark in *Lardner's Algebraic Geometry*, No. 70, it would seem that the author was under the impression that these points are always identical. It is evident, however, that this is not generally the case, except when the figure is a triangle.

The above propositions occurred to me some years ago, when I was engaged in drawing up a course of lectures on statics. I have found that equations (2) are not new, but the rest, as far as I know, have never been published. The mode of investigation seems to be as simple as could be desired; and, as the formulas are of some practical importance, I should think that they might be interesting to the junior readers of the *Mathematician*.

Manchester, Dec. 4th, 1846.

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ON THE DETERMINATION OF THE NATURE OF SURFACES OF THE SECOND DEGREE.

[*James Cockle, Esq., M.A., Barrister-at-Law.*]

Let $U = x^2 + ay^2 + cxy + Z'_1 x + Z'_2 y + Z' = 0$

be the equation of surfaces of the second degree and let

$$U = VW$$

where
$$V = x + a_1 y + s_1$$
$$W = x + a' y + s'$$

then, $a_1 a' = a$, and $a_1 + a' = c$

so that a_1 and a' are the roots of the quadratic

$$a^2 - ca + a = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and a_1 and a' have each but one value.

Again,
$$s_1 + s' = Z'_1$$
$$a' s_1 + a_1 s' = Z'_2$$

hence $(a' - a_1)\, s' = a' Z'_1 - Z'_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$

and $(a_1 - a')\, s_1 = a_1 Z'_1 - Z'_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$

multiply (2) by (3), then, since $(a_1 - a')^2 = (a_1 + a')^2 - 4a_1 a'$

and $s_1 s' = Z''$

the result will be, after proper substitutions,

$$(c^2 - 4a)\, Z'' + a Z'^2_1 - c Z'_1 Z'_2 + Z'^2_2 = 0 \quad \ldots\ldots\ldots\ldots (4)$$

Now, (the coordinates being x, y, and z,) Z'_1 and Z'_2 are linear, and Z'' is a quadratic function of z; so that (4) is a quadratic equation in z.

The equations (1) and (4) serve to determine completely the nature of the surface represented by our introductory equation. But my object in this paper is, not to discuss those equations, but to illustrate by an example, (1) a very simple way of determining a_1 and a' (which have each only one value) and s_1 and s' (which may each have one, two, or an infinite number of values), and (2) the inferences to be drawn when those quantities are determined. In the following process the reader will recognize a method which I gave at p. 114 of vol. I. of the *Mathematician*; the example itself is a particular case of my equation (a) at p. 344 of vol. XLV. of the *Mechanics' Magazine*.

Example.—What is the surface represented by the equation

$$x^2 + x - y^2 + (2z+1)y + z^2 + z - 4 = 0\ ?$$

Add and subtract $\frac{1}{4}$; then we have

$$(x + \tfrac{1}{2})^2 - y^2 + (2z+1)y + z^2 + z - 4\tfrac{1}{4} = 0$$

Add and subtract $\frac{1}{4}(2z+1)^2$, the square of half the coefficient of y; then this last equation becomes

$$(x + \tfrac{1}{2})^2 - (y - z - \tfrac{1}{2})^2 + 2z^2 + 2z - 4 = 0 \quad \ldots\ldots \quad \ldots\ldots (5)$$

Let $2z^2 + 2z - 4 = 0$; therefore $z = 1$ or -2, and (5) becomes, on decomposing the first two terms into factors,

$$(x - y + z + 1)(x + y - z) = 0.$$

Hence, since this last equation indicates a system of two straight lines, we see that the surface in question is one which admits of rectilinear intersection

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with two and only two planes parallel to a coordinate plane. The surface represented by the given equation is, consequently, the *hyperboloid of one sheet*.

The hyperboloid of one sheet has rectilinear intersection either with *two* planes parallel to a coordinate plane, or with *none*—the hyperbolic paraboloid with either *one* such plane or with none. This case must be distinguished from that in which the *two* planes mentioned in the first case coincide—when this last obtains the given surface is a *cone* and the above equation in z would have equal roots; while in the case of the hyperbolic paraboloid it has essentially but one root. In the term "plane parallel to a coordinate plane" I include the coordinate plane itself; *i.e.*, the case in which z is zero.

The discussion in the last paragraph is not to be considered as complete. For instance we have not pointed out (in connection with the method of this paper) to *how many* of the coordinate planes the rectilinear sections may be parallel in the case of each particular surface. On commencing the operations in the above example with z instead of x, (x being isolated, and determined separately), it will be seen that there are in that case *two* such planes; and there are only two, for there are no real rectilinear intersections parallel to the plane of xz—as will be seen on isolating y.

I might now proceed to discuss the cases in which a_1, a', s_1, s', z, are (all or some) unreal. For instance, when the planes represented by $V = 0$ and $W = 0$ are real, and the values of z are unreal and unequal, the surface is the *hyperboloid of two sheets*.

The subject of this paper is connected with that of a question which I have proposed for solution in the *Lady's and Gentleman's Diary* for 1847. As the correspondents of that valuable work will probably give results which (in substance at least) will be equivalent to the development of (4), I shall say nothing upon that point, but merely suggest the desirableness of obtaining, if possible, some convenient working formulæ for the discrimination of the nature of surfaces of the second degree. Nor shall I do more than allude to the fact that in certain cases our process becomes inapplicable, for instance when the *squares* disappear.

ON THE EQUATIONS TO SURFACES OF THE SECOND ORDER WHICH TOUCH THREE GIVEN PLANES, OR WHICH ARE INSCRIBED IN A GIVEN TETRAHEDRON OR PARALLELOPIPED.

[*Mr. Thomas Weddle, Newcastle-upon-Tyne.*]

We must first ascertain the form of the equation of the second degree between two variables, u and v, so that it may be equivalent to the sum or difference of two squares.

Let this equation be

$$A^2u^2 + B^2v^2 + AB luv + Amu + Bnv + 1 = 0 \ldots\ldots\ldots (1)$$

which is the same as

$$\{Au + \tfrac{1}{2}Blv + \tfrac{1}{2}m\}^2 + B^2(1 - \tfrac{1}{4}l^2)v^2 + B(n - \tfrac{1}{2}lm)v + (1 - \tfrac{1}{4}m^2) = 0.$$

Hence (1) will be resolvable into the sum or difference of two squares providing

$$4B^2(1 - \tfrac{1}{4}l^2)(1 - \tfrac{1}{4}m^2) = B^2(n - \tfrac{1}{2}lm)^2;$$
$$\therefore l = \tfrac{1}{2}mn \pm \tfrac{1}{2}\sqrt{(m^2 - 4)(n^2 - 4)}.$$

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Assume $m=\alpha+\alpha^{-1}$ and $n=\beta+\beta^{-1}$, then will

$$l=\tfrac{1}{2}(\alpha+\alpha^{-1})(\beta+\beta^{-1})\pm\tfrac{1}{2}(\alpha-\alpha^{-1})(\beta-\beta^{-1})$$
$$=\alpha\beta+\alpha^{-1}\beta^{-1} \text{ or } \alpha\beta^{-1}+\alpha^{-1}\beta.$$

Hence (1) becomes

$$A^2u^2+B^2v^2+AB(\alpha\beta^{\pm1}+\alpha^{-1}\beta^{\mp1})uv+A(\alpha+\alpha^{-1})u+B(\beta+\beta^{-1})v+1=0 \quad \ldots (2)$$

Let $\alpha+\alpha^{-1}=2\cos\theta$, and $\beta+\beta^{-1}=2\cos\phi$, then will $\alpha\beta^{\pm1}+\alpha^{-1}\beta^{\mp1}=2\cos(\theta\pm\phi)$, and (2) takes the equivalent form

$$A^2u^2+B^2v^2+2AB\cos(\theta\pm\phi).\ uv+2A\cos\theta.\ u+2B\cos\phi.\ v+1=0 \ldots (2')$$

If we restrict θ and ϕ to be real, and α and β to be free from imaginary exponentials, (2) will consist of the difference and (2′) of the sum of two squares. It must also be observed that if the co-efficients of u^2 and v^2 in (2) have different signs, that is, if B^2 be negative, then β^2 must be negative, otherwise some of the co-efficients of (2) would be imaginary.

Let it now be required to find the equation to surfaces of the second order touching the three given planes

$$u=0, \qquad v=0, \qquad w=0 \ldots\ldots\ldots\ldots (3).$$

Assume the required equation to be,

$$A^2u^2+B^2v^2+C^2w^2+ABluv+ACmuw+BCnvw+Apu+Bqv+Crw+1=0 \quad \ldots\ldots (4)$$

where for convenience the co-efficients of u^2, v^2 and w^2 are written as squares; but it must be borne in mind that any of the quantities A, B, C may be of the form $K\sqrt{-1}$.

If we put $w=0$, in (4) we shall have

$$A^2u^2+B^2v^2+ABluv+Apu+Bqv+1=0 \ldots\ldots\ldots\ldots (5)$$

Now since $w=0$ is the equation to a tangent plane, this plane will, in general, meet the surface in one point only, and hence (5) will be resolvable into the sum of two squares. In the case, however, of the hyperboloid of one sheet and the hyperbolic paraboloid, the tangent plane intersects the surface in two straight lines, and hence (5) must now be equivalent to the difference of two squares. With respect to cones and cylinders, the tangent plane touches along a straight line, and hence (5) must, in this case, be a complete square, or, which is the same thing, one of the squares into which (5) is conceived to be resolvable must vanish. The equation (5) must consequently be of the form (2).

$$\therefore\ l=\alpha\beta+\alpha^{-1}\beta^{-1}, \quad p=\alpha+\alpha^{-1}, \quad \text{and } q=\beta+\beta^{-1}.$$

Similarly putting $v=0$, we get

$$m=\alpha\gamma+\alpha^{-1}\gamma^{-1}, \quad p=\alpha+\alpha^{-1}, \quad \text{and } r=\gamma+\gamma^{-1},$$

and putting $u=0$, we have

$$n=\beta\gamma^{\pm1}+\beta^{-1}\gamma^{\mp1}, \quad q=\beta+\beta^{-1}, \quad \text{and } r=\gamma+\gamma^{-1}.$$

It will be observed that the double signs have been omitted in the expressions for two of three quantities, l, m, n. A little consideration will make it evident that we lose nothing in generality by so doing, recollecting that the omitted values of l and m may be transformed into the others by writing β for β^{-1} and γ for γ^{-1}, while the expressions for q and r suffer no essential

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alteration. Moreover one of the values of n, namely, $n = \beta\gamma^{-1} + \beta^{-1}\gamma$, must be rejected, for if it be adopted, (4) will be found to be equivalent to the sum or difference of two squares, and it will consequently denote (not a surface of the second order, but) a straight line or two planes. We must therefore take $n = \beta\gamma + \beta^{-1}\gamma^{-1}$, and hence the equation to surfaces of the second order touching the three planes (3) may be written

$$\begin{aligned} A^2u^2+B^2v^2+C^2w^2+AB(a\beta+a^{-1}\beta^{-1})uv+AC(a\gamma+a^{-1}\gamma^{-1})\,uw \\ +BC(\beta\gamma+\beta^{-1}\gamma^{-1})vw+A(a+a^{-1})u \\ +B(\beta+\beta^{-1})v+C(\gamma+\gamma^{-1})w+1=0 \ldots\ldots\ldots (6) \end{aligned}$$

or, which is the same thing,

$$\begin{aligned} A^2u^2+B^2v^2+C^2w^2+2AB\cos(\theta+\phi)uv+2AC\cos(\theta+\Psi)uw+2BC\cos(\phi+\Psi)vw \\ +2A\cos\theta.\,u+2B\cos\phi.\,v+2C\cos\Psi.\,w+1=0\ldots\ldots(6') \end{aligned}$$

Either (6) or (6′) is the equation to all the surfaces of the second order touching the given planes; but if we restrict a^2, β^2, γ^2, θ, ϕ and Ψ to be real quantities, then it appears from what has been previously said that (6) will be the equation to hyperboloids of one sheet, and hyperbolic paraboloids touching the given planes, while (6′) will be the equation to the remaining surfaces of the second order. In the former case it must be remembered that a^2, β^2 and γ^2 must have respectively the same signs as A^2, B^2 and C^2. The equation to the cones (or cylinders, as the case may be,) touching the given planes is included either in (6) or (6′) and it might be shown that the equation in this case is

$$A^2u^2+B^2v^2+C^2w^2-2ABuv-2ACuw-2BCvw=0^*\ldots\ldots\ldots(6'').$$

We shall next proceed to investigate the equation to surfaces of the second order which touch four given planes, that is, which are inscribed in a given tetrahedron. Let the equations to the faces of the tetrahedron be

$$t=0, \quad u=0, \quad v=0, \quad w=0\ldots\ldots\ldots\ldots\ldots(7)$$

In (2) and (2′) substitute $A \div C$, $B \div C$, $u \div w$, and $v \div w$ for A, B, u and v, and multiply by C^2w^2; hence the equations

$$\begin{aligned} A^2u^2+B^2v^2+C^2w^2+AB(a\beta^{\pm1}+a^{-1}\beta^{\pm1})uv+AC(a+a^{-1})uw \\ +BC(\beta+\beta^{-1})vw=0\ldots.(8), \end{aligned}$$

and

$$\begin{aligned} A^2u^2+B^2v^2+C^2w^2+2AB\cos(\theta\pm\phi)uv+2AC\cos\theta.uw+2BC\cos\phi.vw=0..(8'), \end{aligned}$$

are resolvable into the sum or difference of two squares.

Assume then,

$$\begin{aligned} A^2t^2+B^2u^2+C^2v^2+D^2w^2+ABltu+ACmtv+BCnuv+ADptw \\ +BDquw+CDrvw=0\ldots\ldots\ldots (9) \end{aligned}$$

for the equation to the surfaces touching the planes (7). Put t, u, and v successively $= 0$, and, with the aid of (8), we shall find by the same reasoning as before, $l = \beta\gamma + \beta\gamma^{-1}$, $m = a\gamma + a^{-1}\gamma^{-1}$, $n = a\beta + a^{-1}\beta^{-1}$, $p = \gamma + \gamma^{-1}$, $q = \beta + \beta^{-1}$, and $r = a + a^{-1}$. Now put w also $= 0$, and substitute the values just found; hence the equation

$$\begin{aligned} A^2t^2+B^2u^2+C^2v^2+AB(\beta\gamma+\beta^{-1}\gamma^{-1})tu+AC(a\gamma+a^{-1}\gamma^{-1})tv \\ +BC(a\beta+a^{-1}\beta^{-1})uv=0 \end{aligned}$$

must be resolvable into the sum or difference of two squares. There must

* See also "Mathematician," vol. I, p. 151, equation (14).

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therefore (8) be two quantities λ and μ such that $\lambda\mu+\lambda^{-1}\mu^{-1}=\beta\gamma+\beta^{-1}\gamma^{-1}$, $\lambda+\lambda^{-1}=a\gamma+a^{-1}\gamma^{-1}$, and $\mu+\mu^{-1}=a\beta+a^{-1}\beta^{-1}$ (we need not here write $\lambda\mu\pm1+\lambda^{-1}\mu^{\mp1}$, for μ may be changed into μ^{-1}); hence, solving these quadratics, $\lambda\mu=\beta\gamma$ or $\beta^{-1}\gamma^{-1}$, $\lambda=a\gamma$ or $a^{-1}\gamma^{-1}$, and $\mu=a\beta$ or $a^{-1}\beta^{-1}$; multiply the values of λ and μ, $\therefore\lambda\mu=a^2\beta\gamma$, or $a^{-2}\beta^{-1}\gamma^{-1}$, or $\beta\gamma^{-1}$, or $\beta^{-1}\gamma$; equate the values of $\lambda\mu$, hence $a\beta\gamma=\pm1$, or $a=\pm1$, or $\beta=\pm1$, or $\gamma=\pm1$; but all these relations except the first must be rejected, otherwise (9) would be equivalent to the sum or difference of two squares; moreover we need attend only to the absolute values of a, β, and γ, since changing the sign of a, for instance, is evidently the same as changing that of C. We may therefore take $a\beta\gamma=1$, and the preceding values of l, m, and n become $l=a+a^{-1}$, $m=\beta+\beta^{-1}$, and $n=\gamma+\gamma^{-1}$. Hence the equation (9) to surfaces of the second order which are inscribed in the given tetrahedron or rather (to speak more correctly) which touch its faces, reduces to

$$A^2t^2+B^2u^2+C^2v^2+D^2w^2+(ABtu+CDvw)(a+a^{-1})$$
$$+(ACtv+BDuw)(\beta+\beta^{-1})+(ADtw+BCuv)(\gamma+\gamma^{-1})=0\ldots.(10),$$

with the condition
$$a\beta\gamma=1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Or,
$$A^2t^2+B^2u^2+C^2v^2+D^2w^2+2(ABtu+CDvw)\cos\theta$$
$$+2(ACtv+BDuw)\cos\phi+2(ADtw+BCuv)\cos\Psi=0\ldots(10'),$$

with the condition
$$\theta+\phi+\Psi=2\pi\ldots\ldots\ldots\ldots\ldots\ldots(11')$$

Remarks similar to those that were made upon (6) and (6′) will apply to (10) and (10′).

One of the quantities a, β, γ might easily be eliminated from (10), and one of θ, ϕ, Ψ from (10′), but I prefer leaving the equations as they are, on account of the superior symmetry of the present forms. For the same reason I have introduced four arbitrary quantities A, B, C, D, though I might have assumed any of them equal to unity.

Let us now endeavour to find the equation to surfaces of the second order touching the faces of a given parallelopiped.

Taking the points in which the diagonals intersect for the origin, we may denote the equations to three contiguous faces by

$$u-a=0,\qquad v-b=0,\qquad w-c=0\ldots\ldots\ldots\ldots(12)$$

and the faces opposite to these by

$$u+a=0,\qquad v+b=0,\qquad w+c=0\ldots\ldots\ldots\ldots(13),$$

where u, v, and w are each of the form $px+qy+rz$, and they will be equal to x, y, and z respectively, when the coordinate planes are parallel to the faces of the parallelopiped.

Substituting $u-a$, $v-b$, and $w-c$ for u, v and w in (6), we shall have the equation to the surfaces of the second order touching the faces (12). Expand, equate the coefficients of u, v and w to zero, and solve the resulting equations for A, B, and C. Hence $2Aa(a\beta\gamma-a^{-1}\beta^{-1}\gamma^{-1})=\beta\gamma-\beta^{-1}\gamma^{-1}$, $2Bb(a\beta\gamma-a^{-1}\beta^{-1}\gamma^{-1})=a\gamma-a^{-1}\gamma^{-1}$, and likewise

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$2Cc(\alpha\beta\gamma - \alpha^{-1}\beta^{-1}\gamma^{-1}) = \alpha\beta - \alpha^{-1}\beta^{-1}$. Eliminate A, B, C from (6) by means of these equations,

$$\therefore \quad (\beta\gamma - \beta^{-1}\gamma^{-1})^2 \cdot \left(\frac{u}{a}\right)^2 + (\alpha\gamma - \alpha^{-1}\gamma^{-1})^2 \cdot \left(\frac{v}{b}\right)^2 + (\alpha\beta - \alpha^{-1}\beta^{-1})^2 \cdot \left(\frac{w}{c}\right)^2$$

$$+ (\alpha\beta + \alpha^{-1}\beta^{-1})(\alpha\gamma - \alpha^{-1}\gamma^{-1})(\beta\gamma - \beta^{-1}\gamma^{-1}) \cdot \frac{uv}{ab}$$

$$+ (\alpha\gamma + \alpha^{-1}\gamma^{-1})(\beta\gamma - \beta^{-1}\gamma^{-1})(\alpha\beta - \alpha^{-1}\beta^{-1}) \cdot \frac{uw}{ac}$$

$$+ (\beta\gamma + \beta^{-1}\gamma^{-1})(\alpha\beta - \alpha^{-1}\beta^{-1})(\alpha\gamma - \alpha^{-1}\gamma^{-1}) \cdot \frac{vw}{bc}$$

$$+ (\alpha\beta\gamma - \alpha^{-1}\beta^{-1}\gamma^{-1})(\alpha - \alpha^{-1})(\beta - \beta^{-1})(\gamma - \gamma^{-1}) = 0.$$

This equation may be modified by writing α, β, and γ for $\beta\gamma$, $\alpha\gamma$, and $\alpha\beta$ respectively, and we thus get,

$$(\alpha - \alpha^{-1})^2 \cdot \left(\frac{u}{a}\right)^2 + (\beta - \beta^{-1})^2 \cdot \left(\frac{v}{b}\right)^2 + (\gamma - \gamma^{-1})^2 \cdot \left(\frac{w}{c}\right)^2$$

$$+ (\gamma + \gamma^{-1})(\beta - \beta^{-1})(\alpha - \alpha^{-1}) \cdot \frac{uv}{ab}$$

$$+ (\beta + \beta^{-1})(\alpha - \alpha^{-1})(\gamma - \gamma^{-1}) \cdot \frac{uw}{ac} + (\alpha + \alpha^{-1})(\gamma - \gamma^{-1})(\beta - \beta^{-1}) \cdot \frac{vw}{bc}$$

$$+ (\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}} - \alpha^{-\frac{1}{2}}\beta^{-\frac{1}{2}}\gamma^{-\frac{1}{2}}) \cdot (\alpha^{-\frac{1}{2}}\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}} - \alpha^{\frac{1}{2}}\beta^{-\frac{1}{2}}\gamma^{-\frac{1}{2}}) \cdot (\alpha^{\frac{1}{2}}\beta^{-\frac{1}{2}}\gamma^{\frac{1}{2}} - \alpha^{-\frac{1}{2}}\beta^{\frac{1}{2}}\gamma^{-\frac{1}{2}})$$

$$\times (\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\gamma^{-\frac{1}{2}} - \alpha^{-\frac{1}{2}}\beta^{-\frac{1}{2}}\gamma^{\frac{1}{2}}) = 0 \ldots \ldots \ldots \ldots (14)$$

Or,

$$\sin^2\theta \cdot \left(\frac{u}{a}\right)^2 + \sin^2\phi \cdot \left(\frac{v}{b}\right)^2 + \sin^2\Psi \cdot \left(\frac{w}{c}\right)^2 + 2\cos\Psi \sin\phi \sin\theta \cdot \frac{uv}{ab}$$

$$+ 2\cos\theta \sin\phi \sin\Psi \cdot \frac{uw}{ac} + 2\cos\theta \sin\Psi \sin\phi \cdot \frac{vw}{bc}$$

$$- 4\sin\tfrac{1}{2}(\theta + \phi + \Psi) \sin\tfrac{1}{2}(-\theta + \phi + \Psi) \sin\tfrac{1}{2}(\theta - \phi + \Psi) \sin\tfrac{1}{2}(\theta + \phi - \Psi) = 0$$

$$\ldots . (14').$$

Now if we had proceeded with (6) and (13) in the same way as we have with (6) and (12), we should have got the same equation. Hence (14) or (14′) is the equation to surfaces of the second order inscribed in (or rather, touching the faces of,) the given parallelopiped. Since u, v and w are each of the form $px + qy + rz$, (14) or (14′) denotes surfaces of the second order having their centres at the origin of coordinates, and hence the centres of all the surfaces of the second order touching the faces of a parallelopiped is a fixed point which coincides with the intersection of the diagonals of the parallelopiped. See "Mathematician," Exercise 90.

Newcastle-upon-Tyne,
August 25th, 1846.

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SOLUTIONS OF MATHEMATICAL EXERCISES.

[Besides the solutions of Messrs. Hearn and Weddle of Exercise 87, given in our last number (Mr. Weddle's is given at page 182), a correct and elegant solution was also received from Mr. R. Harley, Northwich, Cheshire.]

LXXXVIII. *Mr. Fenwick.*

The equation in general of a surface of the second order when referred to any three coordinate planes, involves nine independent coefficients; if, however, the planes of reference be tangential to the surface, three of the nine coefficients will be functions of the remaining six. It is required to express the equation of the surface in terms of the six coefficients.

[Solution. *Mr. G. W. Hearn.*]

The equation to a surface of the second order touched by three given planes $u = 0$, $v = 0$, $w = 0$, is

$$A^2u^2 + B^2v^2 + C^2w^2 - 2ABuv - 2ACuw - BCvw = \{ax + by + cz + 1\}^2.$$

For suppose in this equation one of the three u, v, w, as for example, u is zero, the equation reduces to

$$(Bv - Cw)^2 = \{ax + by + cz + 1\}^2,$$

or

$$Bv - Cw = \pm\{ax + by + cz + 1\},$$

which together with the equation $u = 0$, will either represent two straight lines, or a point, the former when A, B, C are real, the latter when they are of the form $a\sqrt{-1}$.

In the former case, the surface is the hyperboloid of one sheet (including the cone), in the latter, any of the other surfaces of the second order.

That the plane $u = 0$ touches the surface is manifest from the consideration that when the equation $u = 0$ is combined with the equation of the surface, it gives either a point or two straight lines.

It seems necessary to say a few words in addition as to the case of the cone.

We must here consider A, B, C infinitely great compared with a, b, c or 1, so that the equation becomes

$$A^2u^2 + B^2v^2 + C^2w^2 - 2ABuv - 2ACuw - 2BCvw = 0,$$

which being homogeneous in u, v and w, is evidently the equation to a cone whose vertex is determined by $u = 0$, $v = 0$, $w = 0$. Also u, v, and w are tangential planes, for putting $u = 0$, the equation is

$$Bv - Cw = 0;$$

so that $u = 0$, and $Bv - Cw = 0$, must represent a straight line, that is, $u = 0$ intersects the cone in one straight line, and therefore touches it.

It must be observed, that in the foregoing I have not restricted myself to the condition that the planes of reference shall be the tangent planes. Introducing this condition we have for the equation

$$Ax^2 + By^2 + Cz^2 - 2ABxy - 2ACxz - 2BCyz = \{ax + by + cz + 1\}^2,$$

where A, B, C, a, b, c are the six arbitrary constants.

Good solutions were received from Messrs. James Reid, Glasgow; and T. Weddle, Newcastle. The proposer's solution is given at page 177 of our last number.

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LXXXIX. *Mr. Thomas Weddle.*

Sum the series

$$\operatorname{cosec}^{-1}\sqrt{[(1^2+1)(2^2+1)]}+\operatorname{cosec}^{-1}\sqrt{[(2^2+1)(3^2+1)]}\ldots\ldots+\operatorname{cosec}^{-1}\sqrt{[(n^2+1)\{(n+1)^2+1\}]};$$

also find a series, the sum of which ad infinitum, shall be a square, the sum of n terms a square, and the sum (ad inf.) of the series remaining, after rejecting the n terms, also a square.

[SOLUTION. *Mr. Thomas Weddle, the proposer; and similarly by Mr. R. Harley, Northwich.*]

Let

$$\operatorname{cosec}^{-1}\sqrt{[(n^2+1)\{(n+1)^2+1\}]}=\theta_n,\ \therefore (n^2+1)\{(n+1)^2+1\}=\operatorname{cosec}^2\theta_n,$$

and

$$\frac{2}{(n^2+1)\{(n+1)^2+1\}}=2\sin^2\theta_n=1-\cos 2\theta_n,$$

or

$$\cos 2\theta_n=1-\frac{2}{(n^2+1)\{(n+1)^2+1\}}.$$

Now if $a_{n-1}=\cos^{-1}\dfrac{2n}{n^2+1}=\sin^{-1}\dfrac{n^2-1}{n^2+1}$, then

$$\cos(a_n-a_{n-1})=\cos a_n\cos a_{n-1}+\sin a_n\sin a_{n-1}=\frac{n^4+2n^3+3n^2+2n}{(n^2+1)\{(n+1)^2+1\}}$$

$$=1-\frac{2}{(n^2+1)\{(n+1)^2+1\}};$$

$$\therefore\quad \cos 2\theta_n=\cos(a_n-\cos a_{n-1}).$$

Whence $\theta_n=\frac{1}{2}a_n-\frac{1}{2}a_{n-1}=\frac{1}{2}\cos^{-1}\dfrac{2(n+1)}{(n+1)^2+1}-\frac{1}{2}\cos^{-1}\dfrac{2n}{n^2+1}$;

hence if S_n represent the sum of n terms, $\Delta S_n=\theta_{n+1}=\frac{1}{2}\Delta.\cos^{-1}\dfrac{2(n+1)}{(n+1)^2+1}$,

$$\therefore\quad S_n=\tfrac{1}{2}\cos^{-1}\frac{2(n+1)}{(n+1)^2+1}+C.$$

Now $\frac{1}{2}\cos^{-1}\dfrac{4}{5}+C=S_1=\operatorname{cosec}^{-1}\sqrt{(1^2+1)(2^2+1)}=\frac{1}{2}\cos^{-1}\dfrac{4}{5}-\cos^{-1}1$

$$=\tfrac{1}{2}\cos^{-1}\frac{4}{5},\quad \therefore C=0.\quad \text{Hence } S_n=\tfrac{1}{2}\cos^{-1}\frac{2(n+1)}{(n+1)^2+1},$$

or $\frac{1}{2}\sin^{-1}\dfrac{n^2+2n}{(n+1)^2+1}$, or $\dfrac{\pi}{4}-\operatorname{cosec}^{-1}\sqrt{\{(n+1)^2+1\}}$. And also, $S_\infty=\dfrac{\pi}{4}$.

Again, for the second part of the question, let S^2_n be the sum of n terms, and $S^2_{\infty-n}$, the sum of the remaining series, then we have

$$S^2_\infty=S^2_n+S^2_{\infty-n}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and the series is $S^2_1+(S^2_2-S^2_1)+(S^2_3-S^2_2)+(S^2_4-S^2_3)+\ldots=S^2_\infty\ldots(2)$; the general term being $\phi(n)=S^2_n-S^2_{n-1}$; hence if the first term follows the same law as the others, we must have, $\phi(1)$ or $S^2_1=S^2_1-S^2_0$,

$$\therefore\quad S_0=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Now we know that

$$1^2=\left(\frac{n^2-r^2}{n^2+r^2}\right)^2+\left(\frac{2rn}{n^2+r^2}\right)^2\ldots\ldots\ldots\ldots(4).$$

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Hence, if we take $S_\infty = 1$, S_n will be $= 1$, when n is infinite; we must consequently take $S_n = \frac{n^2-r^2}{n^2+r^2}$, and not $= \frac{2rn}{n^2+r^2}$, for the former $=1$, and the latter $= 0$, when $n = \infty$; but we must observe that (3) would not be satisfied by $S_n = \frac{n^2-r^2}{n^2+r^2}$, which vanishes for $n=r$, and not for $n=0$; substitute therefore $n+r$ in (4) instead of n, and we have

$$1^2 = \left(\frac{n^2+2rn}{(n+r)^2+r^2}\right)^2 + \left(\frac{2r(n+r)}{(n+r)^2+r^2}\right)^2;$$

we shall therefore take $S_n = \frac{n^2+2rn}{(n+r)^2+r^2}$, which vanishes for $n=0$,

$$\therefore S_{\infty-n} = \frac{2r(n+r)}{(n+r)^2+r^2}, \text{ and } \phi(n) = S^2{}_n - S^2{}_{n-1} = \left(\frac{n^2+2nr}{(n+r)^2+r^2}\right)^2 - \left(\frac{(n-1)^2+2r(n-1)}{(n-1+r)^2+r^2}\right)^2;$$

this being substituted in (2), will give a series satisfying the proposed conditions.

For example, let $r=1$, then $\phi(n) = \frac{4(2n+1)(n^2+n-1)(n^2+n+1)}{(n^2+1)^2\{(n+1)^2+1\}^2}$, but as the constant factor 4 is a square, it may be rejected, hence the series is

$$\frac{3\cdot1\cdot3}{2^2.5^2} + \frac{5.5.7}{5^2.10^2} + \frac{7.11.13}{10^2.17^2} \cdots + \frac{(2n+1)(n^2+n-1)(n^2+n+1)}{(n^2+1)^2\{(n+1)^2+1\}^2} + \cdots$$

the sum of which, ad. inf. is $\frac{1}{4}$, a square; the sum of n terms $= \frac{1}{4}\left(\frac{n^2+2n}{(n+1)^2+1}\right)^2$, a square, and the sum, (ad. inf.) of the remaining series $= \left(\frac{n+1}{(n+1)^2+1}\right)^2$, which is likewise a square.

Good solutions were also sent by Messrs. G. W. Hearn, R. M. College, and Nicholas Smyth, Galway.

XC. *Mr. Fenwick.*

Investigate the locus of the centres of surfaces of the second order inscribed in a given parallelopiped.

[SOLUTION. *Mr. Thomas Weddle; and similarly by Mr. G. W. Hearn.*]

The locus of the middle points of straight lines joining points in two parallel planes is a plane parallel to and equidistant from them. Now if a surface of the second order touch two parallel planes, the straight line joining the points of contact will pass through the centre and be bisected in it. Hence, if surfaces of the second order touch two given parallel planes, the locus of their centres is a plane parallel to and equidistant from them. Applying this property to surfaces of the second order inscribed in a given parallelopiped, we easily see that their centres will be at the same time in three planes passing through the intersections of the diagonals of the parallelopiped and parallel to its faces; that is, the point in which the diagonals of the parallelopiped intersect is the centre of all the surfaces of the second order inscribed in it. Hence the locus spoken of in this exercise degenerates into a point.

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It easily follows, that if four planes A, B, C, D be respectively parallel to other four K, L, M, N, it will, *in general*, be impossible to describe a surface of the second order to touch all these eight planes; for, if this be possible, the intersections of the diagonals of the two parallelopipeds whose faces are A, B, C, K, L, M, and B, C, D, L, M, N respectively, must coincide.

XCI. *Mr. G. W. Hearn, R. M. College.*

A wire ARM bent into the form of a spiral moves in a vertical plane about the pole C, the machine being so counterpoised that C is also the centre of gravity of the revolving mass; CR is a fixed vertical straight rod on which a small ring R slides, the spiral wire also passing through the ring. Find the form of the spiral and other circumstances, that the vertical motion of the ring may be uniform.

[SOLUTION. *By the Proposer.*]

Let M and k be the mass and radius of gyration of the revolving mass, m the mass of the ring, and r the distance CR; then the equation of vis viva gives

$$\left(\mathrm{M}k^2+m\frac{dr^2}{d\theta^2}\right)\frac{d\theta^2}{dt^2}=2mgr+\text{const.}$$

In this making $\frac{dr}{dt}=\mathrm{V}$, we have

$$\mathrm{M}k^2\mathrm{V}^2\frac{d\theta^2}{dr^2}+m\mathrm{V}^2=2mgr+\text{const.}$$

Now supposing the initial values of $\frac{d\theta}{dt}$ and r to be 0 and a respectively, we have

$$\mathrm{M}k^2\mathrm{V}^2\frac{d\theta^2}{dr^2}=2mg(r-a);$$

$$\therefore\quad \mathrm{M}k^2\mathrm{V}^2\theta^2=\frac{8}{9}mg(r-a)^3,$$

making $\theta=0$ when $r=a$.

Put $\frac{9\mathrm{M}k^2\mathrm{V}^2}{8mg}=\mu^3$, and we have

$$r=a+\mu\theta^{\frac{3}{2}},$$

for the equation of the spiral. Also since $r-a=\mathrm{V}t$, we have

$$\theta=\left(\frac{\mathrm{V}t}{\mu}\right)^{\frac{2}{3}},$$

which gives the position at the time t.

Dr. Burns, of Rochester, sent a solution to this exercise.

XCII. *Mr. G. W. Hearn.*

A rough horizontal plane is constrained to revolve round a vertical axis with a constant angular velocity, and a uniform cylinder is placed with its base upon it; if the friction be sufficient to prevent sliding, find the ratio of the diameter of the cylinder to its height, so that it may just upset.

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[SOLUTION. *Mr. Thomas Weddle.*]

Let Q be the point in which the vertical axis meets the horizontal plane and O the centre of the cylinder's base. Through Q and O draw the straight line QBOA meeting the circle in B and A. Assume OA, the radius of the base $= a$, the height of the cylinder $= b$, QO $= c$, the density of the cylinder $= \rho$, the angular velocity $= \omega$, and the accelerating force of gravity $= g$. Take O for the origin of rectangular axes, the axis of z being vertical and that of x coinciding with OA. If $r^2 = (c+x)^2+y^2$, r will be the distance of the particle at (x, y, z) from the axis of revolution, and the centrifugal accelerating force of this particle will be $\omega^2 r$, the components of which parallel to the axes of x and y are $\omega^2(c+x)$ and $\omega^2 y$; the components of the centrifugal moving force acting on the particle at (x, y, z) are consequently $\rho\omega^2(c+x)\,dx\,dy\,dz$ and $\rho\omega^2 y\,dx\,dy\,dz$; but since the latter forces, for the whole cylinder, evidently destroy each other two and two, it is clear that the centrifugal force will tend to turn the cylinder about a line AP drawn through A parallel to the axis of y; taking then the moment of each elemental centrifugal moving force about this line, and summing for the whole cylinder, we have

$$\text{moment about AP} = \rho\omega^2 \iiint (c+x)\,z\,dx\,dy\,dz.$$

The limits of z are 0 and b, those of x, $-x_1 = -\sqrt{a^2-y^2}$, and $x_1 = \sqrt{a^2-y^2}$, and those of y, $-a$ and a. Hence,

$$\begin{aligned}\text{moment about AP} &= \rho\omega^2 \int_0^b z\,dz \int_{-a}^{a}\int_{-x_1}^{x} (c+x)\,dy\,dx \\ &= \tfrac{1}{2}b^2\rho\omega^2 \int_{-a}^{a} 2cx_1\,dy \\ &= b^2 c\rho\omega^2 \int_{-a}^{a} \sqrt{a^2-y^2}\,.\,dy \\ &= \tfrac{1}{2}\pi a^2 b^2 c\rho\omega^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).\end{aligned}$$

Moreover the only force counteracting the centrifugal force is the weight of the body $= \pi a^2 b\rho g$, which may be conceived to act vertically at O. Hence the moment of the weight of the cylinder about AP

$$= \pi a^2 b\rho g \times a = \pi a^2 b\rho g \quad \ldots\ldots\ldots\ldots (2).$$

Now in the case supposed, (1) and (2) are equal, and hence after a little reduction we have

$$\frac{2a}{b} = \frac{c\omega^2}{g} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

or

$$\frac{\text{diameter of cylinder's base}}{\text{height of the cylinder}} = \frac{\text{centrif. accel. force of a particle in the axis of cyl.}}{\text{accelerating force of gravity.}}$$

This result might, however, have been obtained much shorter, thus: The resultant of the centrifugal force of all the particles is horizontal and passes through the centre of gravity of the cylinder, that is, the middle point of the axis; also it is equal to $M\omega^2 c$, M being the mass of the cylinder. Take the moment of this force and of the weight Mg of the body about AP, and we have

$$\tfrac{1}{2}M\omega^2 bc = Mga;$$
$$\therefore\ bc\omega^2 = 2ga$$

the same result as above.

Solutions were given by Dr. Burns, Rochester; and Messrs. J. W. Elliott, Greatham; R. Harley, Northwich; and J. Taylor, Woolwich Common.

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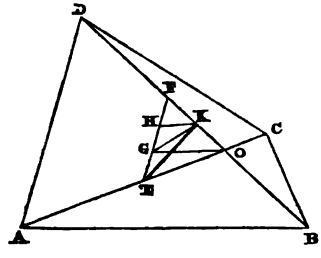
XCIII. *Mr. W. H. Levy, Shalbourne.*

Four times the sum of the squares of the distances of the centre of gravity from the angles of any plane quadrilateral, is equal to the sum of the squares of the four sides and the two diagonals, together with the square of the distance of the centre of gravity from the intersection of the diagonals.

[First Solution. *By W. F., Durham.*]

The following mode of finding the centre of gravity of a quadrilateral ABCD is due to Carnot, and may readily be deduced from either of the methods given in the Lady's Diary for 1833, p. 33.

Take E the middle point of the diagonal AC; then O being the intersection of the diagonals, set off DF equal to BO upon the diagonal DB; join EF and take EG equal to one-third of EF; then G is the centre of gravity of the quadrilateral ABCD.



Bisect GF in H and OF in K, and draw EK, HK, GK.

By a well known theorem,

$$AB^2+BC^2+CD^2+DA^2-4EK^2=AC^2+BD^2 \quad \ldots\ldots (1).$$

But $$4AG^2+4CG^2=8EG^2+8AE^2 \quad \ldots\ldots (2),$$

and $$4BG^2+4DG^2=8GK^2+8BK^2 \quad \ldots\ldots (3).$$

Also $$4HK^2+4EK^2=8EG^2+8GK^2,$$

or $$OG^2+4EK^2=8EG^2+8GK^2 \text{ (since } HK=\tfrac{1}{2}GO) \ldots (4).$$

Adding (2) and (3),

$$4(AG^2+BG^2+CG^2+DG^2)=8EG^2+8GK^2+2AC^2+2BD^2,$$

and substituting from (1) and (4),

$$4(AG^2+BG^2+CG^2+DG^2)=AB^2+BC^2+CD^2+DA^2+AC^2+BD^2+OG^2.$$

[Second Solution. *Messrs. R. T. Ambler, Classical Master at Mr. Paul's School, Shooter's Hill; Nicholas Smyth, of Galway; W. H. Levy, the proposer; and A. D. B., of Charlton Cottage, Blandford.*]

Let O be the intersection of the diagonals, and E and K their points of bisection. Draw EK and bisect it in I* ; draw OI and produce it till IG is equal to one-third of OI: then G is the centre of gravity of the quadrilateral ABCD (Lady's Diary, 1833, p. 33, third solution). Conceive lines to be drawn from G to the angular points A, B, C, D; then

$$4AG^2+4CG^2=8GE^2+8AE^2,$$

$$4BG^2+4DG^2=8GK^2+8BK^2;$$

$$\therefore 4(AG^2+BG^2+CG^2+DG^2)=8AE^2+8BK^2+8GE^2+8GK^2$$

$$=2AC^2+2BD^2+16EI^2+16GI^2$$

$$=2AC^2+2BD^2+4EK^2+OG^2.$$

* The letter I, omitted in the diagram by the engraver, is easily supplied.

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But (Bland's Geometrical Problems, sect. 4, p. 142)

$$AC^2+BD^2=AB^2+BC^2+CD^2+DA^2-4EK^2;$$

whence, substituting in the preceding equality,

$$4(AG^2+BG^2+CG^2+DG^2)=AB^2+BC^2+CD^2+DA^2+AC^2+BD^2+OG^2.$$

Mr. Ambler favoured us with another very neat demonstration.

[THIRD SOLUTION. *Mr. Thomas Weddle, Newcastle-on-Tyne.*]

Let ABCD be the quadrilateral, O the intersection of its diagonals AC and BD, and G its centre of gravity. Take the diagonals for axes and the points A, B, C, D and G by $(-x_1, 0)$, $(0, y_2)$, $(x_3, 0)$, $(0, -y_4)$, and (x_0, y_0); also let $\angle BOC = \omega$. It is easily shown that

$$x_0 = \tfrac{1}{2}(x_3 - x_1), \quad y_0 = \tfrac{1}{2}(y_2 - y_4) \ldots\ldots\ldots\ldots (1).$$

Now

$$GA^2=(x_1+x_0)^2+y_0^2+2\cos\omega.(x_1+x_0)y_0,$$
$$GB^2=x_0^2+(y_2-y_0)^2-2\cos\omega.x_0(y_2-y_0),$$
$$GC^2=(x_3-x_0)^2+y_0^2-2\cos\omega.(x_3-x_0)y_0,$$
$$GD^2=x_0^2+(y_4+y_0)^2+2\cos\omega.x_0(y_4+y_0),$$
$$2GO^2=2x_0^2+2y_0^2+4\cos\omega.x_0y_0;$$

$\therefore GA^2+GB^2+GC^2+GD^2+2GO^2=x_1^2+x_3^2+y_2^2+y_4^2+6x_0^2+6y_0^2-2x_0(x_3-x_1)-2y_0(y_2-y_4)+2\cos\omega.\{6x_0y_0-x_0(y_2-y_4)-y_0(x_3-x_1)\}.$

Reducing this by (1) we have

$$GA^2+GB^2+GC^2+GD^2+2GO^2=x_1^2+x_3^2+y_2^2+y_4^2$$
$$=OA^2+OB^2+OC^2+OD^2\ldots\ldots(2).$$

Also

$$AB^2=x_1^2+y_2^2+2\cos\omega.x_1y_2,$$
$$BC^2=x_3^2+y_2^2-2\cos\omega.x_3y_2,$$
$$CD^2=x_3^2+y_4^2+2\cos\omega.x_3y_4,$$
$$DA^2=x_1^2+y_4^2-2\cos\omega.x_1y_4,$$
$$AC^2=(x_1+x_3)^2, \qquad BD^2=(y_2+y_4)^2,$$
$$9GO^2=9x_0^2+9y_0^2+18\cos\omega.x_0y_0$$
$$=(1)(x_3-x_1)^2+(y_2-y_4)^2+2\cos\omega.(x_3-x_1)(y_2-y_4);$$

$\therefore AB^2+BC^2+CD^2+DA^2+AC^2+BD^2+9GO^2=4\{x_1^2+x_3^2+y_2^2+y_4^2\}$
$=4\{OA^2+OB^2+OC^2+OD^2\}\ldots\ldots\ldots(3).$

Comparing (2) and (3) we have

$$4\{GA^2+GB^2+GC^2+GD^2\}=AB^2+BC^2+CD^2+DA^2+AC^2+BD^2+GO^2,$$

which proves the theorem.

The properties (2) and (3) are also very neat.

Elegant demonstrations were sent by Dr. Burns of Rochester, and Mr. W. Mortimer, of Seedley, near Manchester. Mr. Mortimer deduced from his solution some neat properties, which are reluctantly omitted.

XCIV. *Mr. John Riddle, Nautical School, Greenwich Hospital.*

From the top of a mountain, h miles in height, the visible horizon appeared depressed a°; it is required to shew that, if d be the distance of the boundary of the visible horizon, and D the diameter of the earth,

$$d = h\cot\tfrac{1}{2}a\ldots\ldots(1) \qquad D+h=h\cot^2\tfrac{1}{2}a\ldots\ldots(2).$$

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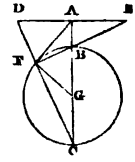
[First Solution. *Mr. John Riddle, the Proposer.*]



Let AB be the height of the mountain, BC the diameter of the earth, angle DAF the dip, and AF the distance of the horizon. Join CF and FB and produce them to meet a line through A at right angles to ABC in D and E. Now angle D is common to the right angled triangles DFE and DAC; therefore angle C = angle E; but (Euc. iii, 22) angle AFE = angle C; therefore angle AFE = angle E; hence angles AFE and AEF are each equal to half angle DAF = $\frac{1}{2}a$, and AF = AE. Again angle D is the complement of angle E, and angle AFD is the complement of angle AFE; hence angle AFD = angle D, and AF = AD.

The right angled triangle ABE, gives

$$AE = AB \cot E; \text{ hence } d = h \cot \tfrac{1}{2}a \quad \ldots\ldots\ldots\ldots (1)$$

Again, the right angled triangle CAD gives

$$AC = AD \cot C; \text{ hence } D + h = d \cot \tfrac{1}{2}a = h \cot^2 \tfrac{1}{2}a \quad \ldots\ldots\ldots (2)$$

Ex. From the top of a mountain 3 miles in height, the visible horizon appeared depressed 2° 13′ 27″; find the diameter of the earth, and the distance of the boundary of the visible horizon.

COMPUTATION.

Log. h =	0·477121	
cot $\frac{1}{2}a$ =	11·711941	add
d = 154·54....	2·189062	
$D + h$ = 7961...	3·901003	
h =	3	
D =	7958	

[Second Solution. *Mr. George F. Lacey, Woolwich Common; and similarly by Mr. H. Hewett, Shooter's Hill.*]

Let G be the centre of the earth, and draw FG; then AFG is a right angled triangle; hence angle AGF = a, and therefore

$$\frac{\frac{D}{2} + h}{d} = \operatorname{cosec} a \ldots\ldots (1), \qquad \frac{\frac{D}{2}}{d} = \cot a \ldots\ldots\ldots (2).$$

Adding and subtracting (1) and (2) gives

$$\frac{D+h}{d} = \operatorname{cosec} a + \cot a = \cot \tfrac{1}{2}a \ldots\ldots\ldots\ldots\ldots (3),$$

$$\frac{h}{d} = \operatorname{cosec} a - \cot a = \tan \tfrac{1}{2}a, \ldots\ldots\ldots\ldots\ldots (4).$$

From (4), $d = h \cot \frac{1}{2}a$.

From (3), $D + h = d \cot \frac{1}{2}a = h \cot^2 \frac{1}{2}a$.

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[THIRD SOLUTION. *Mr. W. Mortimer, Seedley, near Manchester; and in an analogous manner by Messrs. Thomas Wilkinson, Cheapside, Burnley, Lancashire; and W. H. Levy, Shalbourne.*]

Let G be the centre of the earth BFC, AB the mountain, and F the boundary of the visible horizon; then

$$d = \frac{D}{2} \tan a \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

$$\text{and } d^2 = (D+h)\,h, \text{ or, } D = \frac{d^2}{h} - h \ldots\ldots\ldots\ldots (b)$$

Substituting in (a) we have

$$d^2 - 2dh \cot a = h^2,$$

whence $$d - h\cot a = h\sqrt{(1+\cot^2 a)} = h \operatorname{cosec} a,$$

$$\therefore\quad d = h(\cot a + \operatorname{cosec} a) = h \cot \tfrac{1}{2}a \ldots\ldots\ldots (1).$$

Substituting this value in (b) we have

$$D + h = h \cot^2 \tfrac{1}{2}a \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

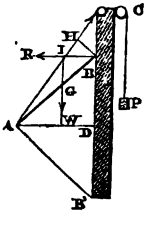
Neat solutions were sent by A.D.B., Charlton Cottage, near Blandford; W. F., Durham; Dr. Burns, Rochester; and Messrs. Ambler, Godfray, Weddle, and Smyth.

XCV. *Mr. R. H. Wright, London.*

An uniform beam whose length is a and weight W, rests with one end against a wall and is supported by a string from the other and passing over a pulley on the top of the wall with a weight P at its extremity; then if the part of the string that subtends the angle that the beam makes with the wall is l, the angle that the beam makes with the horizon ϕ, and h the perpendicular height of the pulley from the lower extremity of the beam when in equilibrium; shew that the inclination of the string to the beam will be

$$\tan^{-1}\left\{\frac{Wl}{2P(a\cos\phi \pm h\tan\phi)}\right\}.$$

[FIRST SOLUTION. *Mr. Weddle, Newcastle-upon-Tyne.*]



Let AB be the beam resting against the wall CD, G its middle point (at which W may be conceived to act parallel to CD), and C the pulley. Draw AD perpendicular to CD; then $AB = a$, $CD = h$, $AC = l$, and $\angle BAD = \phi$; let also $BC = k$, $AD = m$, $\angle BAC = \theta$, and $\angle BCA = \Psi$. By the geometry of the figure we have

$$\theta + \phi + \Psi = \frac{\pi}{2} \ldots\ldots (1), \qquad k = h - a\sin\phi \ldots\ldots\ldots (2),$$

$$m = a\cos\phi \ldots\ldots\ldots (3), \qquad \frac{k}{\sin\theta} = \frac{l}{\cos\phi} \ldots\ldots\ldots (4),$$

$$\cot\Psi = \frac{h}{m} \text{ or } (1), \quad \tan(\theta+\phi) = \frac{h}{m} \ldots\ldots\ldots\ldots (5).$$

From (5) we easily get

$$\tan\theta = \frac{h - m\tan\phi}{m + h\tan\phi} = (3)\ \frac{h - a\sin\phi}{a\cos\phi + h\tan\phi}.$$

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Hence (2), $$\tan\theta = \frac{k}{a\cos\phi + h\tan\phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Moreover, taking the moments of the forces keeping AB in equilibrium about B, we have

$$\mathrm{P}a\sin\theta = \tfrac{1}{2}\mathrm{W}.a\cos\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Comparing (4) and (7), we have

$$2\mathrm{P}.k = \mathrm{W}.l \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Eliminate k between (6, 8),

$$\tan\theta = \frac{\mathrm{W}l}{2\mathrm{P}.(a\cos\phi + h\tan\phi)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

which establishes the exercise.

If the beam were in the position AB′, and the end B′ prevented from sliding down, we might show that

$$\tan\theta = \frac{\mathrm{W}l}{2\mathrm{P}(a\cos\phi - h\tan\phi)},$$

where $\phi = \angle \mathrm{B'AD}$ and $\theta = \angle \mathrm{CAB'}$. Here, however, h denotes the perpendicular height of the pulley from the *upper* extremity of the beam.

[Second Solution. *Mr. H. Godfray, David Place, St. Mark's Road, Jersey.*]

Let AB be the beam, C the pulley, $\mathrm{CD} = h$, $\mathrm{CA} = l$, $\mathrm{AB} = a$, and angle $\mathrm{CAB} = \theta$. Produce the vertical through the middle point G of AB, till it meets AC in I, and join BI. Resolving the forces vertically, we get

$$\mathrm{W} = \mathrm{P}\sin(\theta+\phi) = \mathrm{P}\,\frac{h}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The two forces W and P may be supposed applied at I, and as their resultant must destroy the resistance of the plane, BI must be horizontal; therefore I being the middle point of CA, CB will be equal to BD. Hence

$$\tan(\theta+\phi) = \frac{\mathrm{CD}}{\mathrm{DA}} = 2\,\frac{\mathrm{BD}}{\mathrm{DA}} = 2\tan\phi;$$

and
$$\tan\theta = \tan(\theta+\phi-\phi) = \frac{\tan(\theta+\phi) - \tan\phi}{1+\tan(\theta+\phi)\tan\phi} = \frac{\tan\phi}{1+\frac{h}{\mathrm{AD}}\tan\phi}$$

$$= \frac{\tan\phi}{1+\frac{h}{a\cos\phi}\tan\phi} = \frac{a\sin\phi}{a\cos\phi + h\tan\phi} = \frac{\frac{1}{2}h}{a\cos\phi + h\tan\phi}$$

$$= \frac{\mathrm{P}h}{2\mathrm{P}(a\cos\phi + h\tan\phi)} = (1)\ \frac{\mathrm{W}l}{2\mathrm{P}(a\cos\phi + h\tan\phi)}.$$

[Third Solution. *Mr. W. Mortimer, Seedley, near Manchester.*]

Let angle $\mathrm{ACB} = \alpha$ and angle $\mathrm{CAB} = \theta$; then since $\theta = 90 - (\alpha+\phi)$,

$$\tan\theta = \frac{1-\tan\alpha\tan\phi}{\tan\alpha+\tan\phi} = \frac{h - \mathrm{AD}\tan\phi}{\mathrm{AD}+h\tan\phi} = \frac{\mathrm{BC}}{a\cos\phi + h\tan\phi} \quad \ldots\ldots (1).$$

When the beam is in equilibrium, it is evident that the horizontal line

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BI, and the vertical GI through the centre of gravity of the beam AB meet in I the middle point of AC. Taking the moments about B we have

$$P.a\sin\theta = W.\frac{a}{2}\cos\phi, \text{ or}$$

$$2P : W :: \cos\phi : \sin\theta :: \sin ABC : \sin BAC :: l : BC;$$

hence $$BC = \frac{Wl}{2P}, \text{ and from (1)}$$

$$\tan\theta = \frac{Wl}{2P(a\cos\phi + h\tan\phi)}.$$

[FOURTH SOLUTION. *Mr. George F. Lacey, Woolwich Common.*]

Let angle $BAC = \theta$, and taking moments about B, we have

$$Pa\sin\theta - W\frac{a}{2}\cos\theta = 0,$$

or $$\frac{2P}{W} = \frac{\cos\phi}{\sin\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$$

Again $$\tan(\theta \pm \phi) = \frac{CD}{DA} = \frac{h}{a\cos\theta}; \text{ hence}$$

$$a\cos\phi(\tan\theta \pm \tan\phi) = h(1 \pm \tan\theta\tan\phi),$$

or, $$\tan\theta(a\cos\phi \pm h\tan\phi) = h \pm a\sin\phi = BC \text{ or } B'C \ldots\ldots (\beta).$$

From (α) we have

$$\frac{2P}{W} = \frac{\cos\phi}{\sin\theta} = \frac{\sin ABC}{\sin BAC} = \frac{l}{BC};$$

$$\therefore \quad BC = \frac{Wl}{2P},$$

and substituting this value for BC in (β) we get

$$\tan\theta = \frac{Wl}{2P(a\cos\phi \pm h\tan\phi)}.$$

[FIFTH SOLUTION. *By W. F., Durham; and similarly by Mr. Wright, the Proposer.*]

Let AB be the beam, G its centre of gravity, and let angle $BAC = \beta$; then the equilibrium is maintained by three forces, viz., by W in direction GW; R, the reaction of the wall in BR; and P, the tension of the string in AC.

Now resolving forces vertically, we have

$$P\sin(\beta+\phi) - W = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Taking moments about B,

$$W\cos\phi = 2P\sin\beta \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

By the geometry of the figure

$$\frac{a}{l} = \frac{\cos(\beta+\phi)}{\cos\phi} \ldots (3), \qquad \frac{h}{l} = \sin(\beta+\phi) \ldots\ldots\ldots (4).$$

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Dividing (4) by (3) gives

$$\tan(\beta+\phi) = \frac{h}{a\cos\phi}$$

$$\therefore \quad a\cos\phi(\tan\beta+\tan\phi) = h(1-\tan\beta\tan\phi),$$

$$\text{or,} \quad \tan\beta = \frac{h-a\sin\phi}{a\cos\phi+h\tan\phi} = \frac{BC}{a\cos\phi+h\tan\phi} \quad \ldots\ldots(5).$$

From (2) we obtain

$$W : 2P :: \sin\beta : \cos\phi :: \sin\beta : \sin ABC :: BC : l;$$

$$\therefore \quad BC = \frac{Wl}{2P}, \text{ and hence } (5)$$

$$\tan\beta = \frac{Wl}{2P(a\cos\phi+h\tan\phi)}.$$

The equations (1) and (2) furnish a relation between β and ϕ. From these we have

$$\cos\phi\sin(\beta+\phi) = 2\sin\beta,$$

$$\text{or,} \quad \sin\beta\cos^2\phi+\cos\beta\sin\phi\cos\phi = 2\sin\beta;$$

$$\therefore \quad \tan\beta = \frac{\sin\phi\cos\phi}{2-\cos^2\phi} = \frac{\sin\phi\cos\phi}{1+\sin^2\phi} = \frac{\cos\phi}{\operatorname{cosec}\phi+\sin\phi}.$$

[SIXTH SOLUTION. *Mr. Nicholas Smyth, Galway; and in an analogous manner by Dr. Burns, Rochester.*]

Draw BH perpendicular to AC; then when the equilibrium is maintained, we get

$$W.\frac{AD}{2} = P.BH \quad \ldots\ldots(1).$$

Now $AD = a\cos\phi$; $BD = a\sin\phi$; $BC = h - a\sin\phi$; and the right angled triangles ACD and CBH give AC : AD :: BC : BH, or

$$l : a\cos\phi :: h - a\sin\phi : BH = \frac{a\cos\phi(h-a\sin\phi)}{l};$$

$$\therefore \quad AH = \sqrt{AB^2-BH^2} = \frac{a^2\cos^2\phi+ah\sin\phi}{l};$$

and by substitution (1) becomes

$$\tfrac{1}{2}W.a\cos\phi = \frac{Pa\cos\phi(h-a\sin\phi)}{l}; \quad h-a\sin\phi = \frac{Wl}{2P} \ldots\ldots(2).$$

Again, $\tan BAC = \dfrac{BH}{HA} = \dfrac{a\cos\phi(h-a\sin\phi)}{a^2\cos^2\phi+ah\sin\phi} = (2)\dfrac{Wl}{2P(a\cos\phi+h\tan\phi)}$.

When the beam AB is kept in equilibrium in the position AB', we should have

$$\tan B'AC = \frac{Wl}{2P(a\cos\phi-h\tan\phi)}, \text{ where } h \text{ still denotes CD.}$$

Mr. Ambler favoured us with an excellent solution of this exercise, which gives as the final result,

$$\tan\theta = \frac{Wl}{P(2a\sec\phi+h\tan\phi)}.$$

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This, however, can readily be transformed into the expression obtained in the previous solutions. Thus, in the right angled triangle ABD we get

$$\sin\phi = \frac{BD}{BA} = \frac{\frac{1}{2}h}{a} = \frac{h}{2a}\text{; therefore}$$

$$2a\sin\phi = h, \text{ or } 2a\sin^2\phi = h\sin\phi = h\cos\phi\tan\phi$$

$$\therefore\ 2a(1-\cos^2\phi) = h\cos\phi\tan\phi, \text{ or}$$

$$2a(\sec\phi - \cos\phi) = h\tan\phi;$$

$$\therefore\ 2a\sec\phi + h\tan\phi = 2a\cos\phi + 2h\tan\phi = 2(a\cos\phi + h\tan\phi),$$

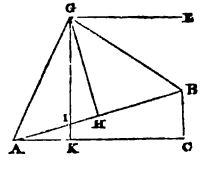
which reduces the expression to the same form as that given in the enunciation of the exercise.

XCVI. *W. Burns, LL.D., St. Margaret's Bank, Rochester.*

Given the radius of a railway curve, the distance between the rails, the height of the outer above the level of the inner rail, and the height of the centre of gravity of the train, to find the velocity of the train when both rails sustain equal parts of the weight.

[First Solution. *Mr. T. Weddle, Newcastle-upon-Tyne; and similarly by Mr. R. T. Ambler, Shooter's Hill.*]

Through the centre of the curve and G the centre of gravity, let a vertical plane be drawn, meeting the inner rail in A and the outer in B; draw AC and GE horizontal, BC and GK vertical, and GH perpendicular to AB. Let r be the radius of the curve, d = AB = distance between the rails, h = BC = height of the outer above the inner rail, and ϕ = > BAC = > HGK; then $\sin\phi = \frac{h}{d}$,



which determines ϕ. Let also M be the mass and v the velocity of the train, and g the accelerating force of gravity. The only forces pressing upon the rails are the weight Mg in the direction GK, and the moving centrifugal force $M\frac{v^2}{r}$ in the direction GE. Now these forces are to be equivalent to equal forces acting in GA and GB, the resultant of which must be in the direction GH bisecting the angle AGB. Hence GH must also be the direction of the resultant of Mg and $M\frac{v^2}{r}$, and consequently the components of these forces perpendicular to GH must be equal and opposite,

$$\therefore\ Mg.\sin\phi = M\frac{v^2}{r}\cos\phi$$

$$\therefore\ v^2 = gr\tan\phi \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which gives the velocity per second, from which that per hour is easily deduced. This result is independent of the height of the centre of gravity.

Since BC will always be very small compared with AB, we may take, $\tan\phi = \sin\phi = \frac{h}{d}$, and hence (1) becomes,

$$v^2 = \frac{ghr}{d} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

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[SECOND SOLUTION. *W. Burns, L.L.D., the Proposer.*]

Let AB $= a$, the distance between the rails,
BC $= b$, the elevation of the outer rail,
GH $= h$, the height of the centre of gravity,
r = radius of the curve, $\pi = 3{\cdot}14159\ldots$, and $g = 32{\cdot}2$ feet.

Now GI the weight of the train when at rest on the curve, is resolved into GH, HI, whereof GH presses equally on A and B, and HI requires to be counterbalanced by the centrifugal force. Also, AC $= \sqrt{a^2 - b^2}$, and by similar triangles, $\sqrt{a^2-b^2} : a :: h : \dfrac{ah}{\sqrt{a^2-b^2}} =$ GI, and $\sqrt{a^2-b^2} : b :: h : \dfrac{bh}{\sqrt{a^2-b^2}} =$ HI. Wherefore, since the centrifugal force : the force of gravity $:: \dfrac{4\pi^2 r}{g} : t^2$,

$$\frac{bh}{\sqrt{a^2-b^2}} : \frac{ah}{\sqrt{a^2-b^2}} :: \frac{4\pi^2 r}{g} : t^2, \text{ and } t = 2\pi \left\{\frac{ar}{bg}\right\}^{\frac{1}{2}}; \text{ hence,}$$

$$v = \frac{2\pi r}{2\pi\left\{\dfrac{ar}{bg}\right\}^{\frac{1}{2}}} = \left\{\frac{bgr}{a}\right\}^{\frac{1}{2}}, \text{ or } v^2 = \frac{gbr}{a}.$$

XCVII. *By Habitancum.*

Determine the values of h and k (these are the only variable quantities) that make the function,

$$u = (a-h)^2(\beta-k)^2 - \{hk + \tfrac{1}{2}a(m+n)(a-2h)\}^2,$$

a maximum or minimum; u, a, β, m, n being *positive*, and h and k subject to the condition

$$mna(a-2h) - \beta(\beta-2k) = 0.$$

[SOLUTION. *Mr. Thomas Weddle.*]

Let $v = (m+n)\dfrac{a}{\beta}$, and $w = \left(\dfrac{1}{m} + \dfrac{1}{n}\right)\dfrac{\beta}{a}$ .....(1);

then $\dfrac{v}{w} = \dfrac{mna^2}{\beta^2}$, and we have (2)

$$u = (a-h)^2(\beta-k)^2 - \{hk + \tfrac{1}{2}v\beta(a-2h)\}^2 \ldots\ldots(2),$$

and, $$v\beta(a-2h) = wa(\beta-2k) \ldots\ldots(3).$$

Assume $$2h = a(1-wz), \qquad 2k = \beta(1-vz) \ldots\ldots(4),$$

then (3) is satisfied, and we have

$$u = \frac{1}{16}a^2\beta^2[(1+vz)^2(1+wz)^2 - \{(1-vz)(1-wz)+2vwz\}^2]$$

$$= \frac{1}{16}a^2\beta^2\{(1+vz)(1+wz) - (1-vz)(1-wz) - 2vwz\}$$
$$\times\{(1+vz)(1+wz) + (1-vz)(1-wz) + 2vwz\}$$

$$= \frac{1}{4}a^2\beta^2\{v+w-vw\}.\{z+vwz^2+vwz^2\} \ldots\ldots(5).$$

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Hence if $$U = z + vwz^2 + vwz^3 \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

U must be a maximum or a minimum. Equate $\frac{dU}{dz}$ to zero, and we have

$$1 + 2vwz + 3vwz^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$\therefore z = -\frac{1}{3}\left\{1 \pm \left(1 - \frac{3}{vw}\right)^{\frac{1}{2}}\right\} \ldots\ldots\ldots\ldots\ldots (8).$$

For brevity, denote these two values of z by $-z_1$, and $-z_2$, $(z_1 > z_2)$. Since v and w are positive and (7) is the limiting equation of (6), we know (since the roots of (6) are easily shown to be real) by the theory of equations, that $z = -z_1$, renders U a maximum and ***positive***, and $z = -z_2$, makes U a minimum and ***negative***.

Moreover, restoring the values of v and w in (5), it is easy to see that

$$u = \frac{1}{4}\alpha\beta\left(\frac{1}{m} + \frac{1}{n}\right)(\beta - m\alpha)(\beta - n\alpha)\,U \ldots\ldots\ldots (9).$$

Now if β be not comprised between the limits $m\alpha$ and $n\alpha$, $(\beta - m\alpha) \times (\beta - n\alpha)$ will be positive, and u and U, having the same sign, will be a maximum or a minimum simultaneously. But since u must be positive, z cannot $= -z_2$, and hence we must have $z = -z_1$, which makes U and therefore u a maximum. Should β however be comprised between the limits $m\alpha$ and $n\alpha$, then $(\beta - m\alpha)(\beta - n\alpha)$ being negative, one of the quantities u, U will be a maximum when the other is a minimum; and u being positive, U must be negative, hence $z = -z_2$ is the only admissible value, and this making U a minimum will render u a maximum. Supposing then $m < n$,

If $\beta < m\alpha$ or $> n\alpha$, $\quad z = -z_1$ makes u a maximum.

If $\beta > m\alpha$ or $< n\alpha$, $\quad z = -z_2$ makes u a maximum.

Hence u does not, under the proposed conditions, admit of a minimum, but has always one maximum.

Knowing the value of z, those of h and k are got from (4).

Solutions to this exercise were also received from Messrs. Ambler, Shooter's Hill; W. Mortimer, Seedley; N. Smyth, Galway; and Dr. Burns, Rochester.

XCVIII. *Pen-and-Ink, Charlton.*

Two points and a straight line, not all in the same plane, being given, it is required to draw two lines from the given points to meet in the given line:

(1) That shall make equal angles with the line;

(2) That shall make the greatest angle with each other.

[Solution. *Mr. Thomas Weddle.*]

Let A and B be the two given points (the student will readily construct the fig.), CD the given line, and first let P be a point in CD such that AP and BP make equal angles with the *opposite* portions of CD. From A and B draw AC and BD perpendicular to CD; let the plane BDC, be turned round CD until BD come into the plane ACD, and let the rotation be such that AC and BD may lie on opposite sides of CD. It is plain that BP will in this revolution describe a right cone; hence AP and B'P (the new position of BP) will make equal angles with the opposite parts of CD,

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and consequently being on different sides of CD, they must be in the same straight line; also B'D is evidently perpendicular to CD. Hence the following construction.

From the given points A and B draw AC and BD perpendicular to the given line CD; in the plane ACD, and on the side of CD opposite to AC, draw B'D perpendicular to CD and = BD; join AB' intersecting CD in P, join also BP; AP and BP will make equal angles with the opposite segments of CD.

If the lines AP and BP are to make equal angles with the *same* part of CD, it might similarly be shown that the construction will be the very same except that B'D must be drawn on the same side of CD as AC.

Next let P' be such a point in CD that the angle AP'B may be a maximum. From A and B draw AC and BC perpendicular to CD, and bisect CD in E. Let AE = a, BE = b, CE = DE = c, EP' = x, and $a^2+b^2-AB^2=2d^2$. Hence $AP'^2=AC^2+CP'^2=AC^2+(c+x)^2=a^2+2cx+x^2$, $BP'^2=BD^2+DP'^2=BD^2+(c-x)^2=b^2-2cx+x^2$,

and $\cos \angle AP'B = \dfrac{AP'^2+BP'^2-AB^2}{2AP'.BP'} = \dfrac{d^2+x^2}{\sqrt{(a^2+2cx+x^2)(b^2-2cx+x^2)}}$ .....(1).

Hence (1) must be a minimum. Equate therefore the differential coefficient of the logarithm of (1) to zero;

$$\therefore \frac{2x}{d^2+x^2} - \frac{c+x}{a^2+2cx+x^2} - \frac{-c+x}{b^2-2cx+x^2} = 0 \text{ .....(2).}$$

On reduction this becomes,

$$(a^2+b^2-4c^2-2d^2)x^3+3(b^2-a^2)cx^2+\left(2a^2b^2+4c^2d^2-(a^2+b^2)d^2\right)x$$
$$-(b^2-a^2)d^2c=0 \text{ .....(3).}$$

From this equation the value of x may be obtained and thence the position of P' is known.

Solutions were also received from Messrs. Ambler, Godfray, and Mortimer.

XXXIX. *Mr. W. S. B. Woolhouse.*

(Proposed in Vol. I., No. 5, p. 272.)

In laying down *fifteen* points upon a plane, they may be so placed that sets of three points ranging exactly in a line, shall exist in *twenty-six* different directions. Required proof.

[Solution. *By the Proposer.*]

An extreme limit may immediately be assigned to the question abstractedly merely by the mode in which the points must necessarily be associated. Each line or range exhibits a triad of points, and amongst all these triads it is evident that any two points cannot appear together more than once, because the repetition of a pair of points would imply a repetition of the same line. The complete number of triads that can be thus collected out of 15 things will comprise the whole of the 105 pairs,* and as each triad involves three distinct pairs, the total number of such triads is 35. We therefore conclude, without reference to geometrical conditions or limitations, that the greatest possible number of ranges cannot exceed the number 35.

It will appear that a general discussion of the problem with the points in space can embrace no more possible ranges than in plano. This is obvious

* See the Lady's and Gentleman's Diary for 1847, page 62, solution to question xii. (1760.)

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by considering the orthographic projection of the figure upon any plane not coincident with planes normal to the several lines. We shall therefore treat the question under its more general aspect.

Three unequal spheres $s_1 s_2 s_3$ being taken ad libitum, both as to magnitude and position, it is well known that cones enveloping each pair have their three vertices ranging in a right line. Let the point which is the vertex of the cone enveloping $s_1 s_2$ be denoted by (1, 2); the vertex of the cone enveloping $s_2 s_3$ by (2, 3), *etc.* Then the three vertices (1, 2), (2, 3), (1, 3) range in a right line, which may be conveniently designated (1, 2, 3). Instead of three let us now take six spheres $s_1 s_2 s_3$ $s_4 s_5 s_6$; then every pair of spheres determining a point, there will be 15 points as in the question. Also the points resulting from every triad of spheres constituting a range, the number of ranges will be 20, being the number of different triads out of six things. We shall therefore have 20 ranges amongst the fifteen points with the six spheres taken ad libitum ..... (α).

It remains to determine how many more ranges can possibly be determined by varying the magnitude and position of one or more of the six spheres. The triads of points furnished by the 20 ranges (α) are distinguished by the repetition of the numerals. Thus (2, 3), (3, 5), in which 3 is repeated, are situated in the line (2, 3, 5) and can therefore only be associated with the point (2, 5) and form one of the twenty ranges already determined. Consequently in putting down new triads of points we must in each of them employ all the six spheres, as no one can be repeated. It is easy therefore to write all the additional triads as follows:—

(16, 23, 45), (16, 24, 35), (16, 25, 34) ..... (A)
(26, 13, 45), (26, 14, 35), (26, 15, 34) ..... (B)
(36, 12, 45), (36, 14, 25), (36, 15, 24) ..... (C)
(46, 12, 35), (46, 13, 25), (46, 15, 23) ..... (D)
(56, 12, 34), (56, 13, 24), (56, 14, 23) ..... (E)

These 15 triads make up the complete set of 35 triads before mentioned, and any one of them may separately be converted into a new range by varying the magnitude and position of one of the spheres. We must now ascertain how many of them can be made to range simultaneously, consistently with the geometrical relations of the figure.

First take the three triads in (A) and let us direct our attention for a a moment to the four spheres $s_2 s_3 s_4 s_5$. As they comprise four different triads the six resulting points will have four ranges and will evidently be the six corners to a complete quadrilateral whose prolonged sides are the four lines (2, 3, 4,), (2, 3, 5), (2, 4, 5,) (3, 4, 5). The three diagonals of this quadrilateral are the lines joining the pairs of points (23, 45), (24, 35), (25, 34). According to (A) the point (1, 6) is required to range in each of these three diagonals, which is impossible, as the three diagonals of a quadrilateral cannot intersect in a common point. The point (1, 6) may however be placed on the intersection of any two of the three diagonals without ranging in the third. We must therefore reject at least one of the triads from each of the sets (A), (B), (C), (D), (E). Let the first triad be cancelled in each of the sets (A), (B, (C), and the ranges suggested by these sets will be

(16, 24, 35), (16, 25, 34)
(26, 14, 35), (26, 15, 34)
(36, 14, 25), (36, 15, 24)
} ..... (β)

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It will be necessary to show that the points (1, 6), (2, 6), (3, 6), thus determined by the first five spheres, are consistent with the previously assigned ranges (12, 16, 26), (13, 16, 36), (23, 26, 36). Let two lines be drawn, the one joining the centre of s_4 with the point (3, 5), and the other joining the centre of s_5 with the point (3, 4). These lines being both situated in the plane passing through the centres of $s_3 s_4 s_5$, will intersect. Let the point in which they intersect be taken for the centre of another sphere s_7, and let its magnitude be such that the point (3, 7) may be situated in the line (3, 4, 5). Then since the point (3, 7) ranges with (3, 4), (4, 7) and also with (3, 4), (3, 5) it is evident that the point (4, 7) must coincide with (3, 5). Similarly the point (5, 7) coincides with (3, 4). Therefore the first and second lines of (β) may be written

(16, 24, 47), (16, 25, 57),
(26, 14, 47), (26, 15, 57).

Hence the points (1, 6), (2, 6) determined by (β) will respectively coincide with (2, 7), (1, 7), and must therefore range in the same line with the point (1, 2). The other two ranges may be similarly exhibited, and thus the points (1, 6), (2, 6), (3, 6) which result from the ranges in (β), are such as to satisfy the previously implied conditions.

The six triads contained in (D), (E) will now, every one of them, be individually excluded. For example take the first in (D) and it will appear that (3, 5) is required to range with (12, 46). But in (β) it already ranges in (16, 24), (26, 14), two of the diagonals of the quadrilateral (1, 2, 4, 6), and it therefore cannot range in the third. The other triads in (D), (E) are all similarly inadmissible.

We have therefore, in all, 26 possible ranges amongst the fifteen points.

For the construction of the diagram the intervention of the spheres will not be required. Any three points (1, 2), (1, 3), (2, 3) being taken in a line, let three lines be drawn respectively through them, those from (1, 2), (1, 3) intersecting in (1, 4), those from (1, 2), (2, 3), in (2, 4), and those from (1, 3), (2, 3) in (3, 4). In like manner, through the same three points as before, let any other three lines be drawn intersecting each other in the points (1, 5), (2, 5), (3, 5). By (β) the points (1, 6), (2, 6), (3, 6) are now determined by the respective intersections of the diagonals therein indicated. The three remaining points are

(4, 5) the mutual intersection of (14, 15), (24, 25), (34, 35),
(4, 6) ,, (14, 16), (24, 26), (34, 36),
(5, 6) ,, (15, 16), (25, 26), (35, 36), (45, 46),

and the three remaining ranges are (12, 16, 26), (13, 16, 36), (23, 26, 36).

Since every four spheres determine a complete quadrilateral the entire figure will contain fifteen complete quadrilaterals, being as many as there are points. These quadrilaterals will have amongst them twenty different sides exhibiting the twenty ranges according to (α), and the other six ranges in (β) will be diagonal ranges involving additional lines.

Solutions were sent by Messrs. Elliott, Hearn, and Weddle.

XCIX. *Septimus Tebay.*

A circular plate, of equal density and thickness, is suspended horizontally by three equal vertical strings attached to three given points in its periphery. If it be twisted horizontally through a very small angle, so that its centre remains in the same vertical line, required the time of an oscillation.

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[First Solution. *By W. F., Durham.*]

Let l = length of each string, m = mass, r = radius of the plate, and θ = the small angle through which it is twisted. Then for small oscillations, the tension of each string will be $\frac{1}{3}mg$, and the resolved part of the tension which is effective in producing motion, will be nearly equal to

$$\tfrac{1}{3}mg.\ \frac{r\theta}{l} = \frac{mgr\theta}{3l}.$$

Hence the moment about the centre of gravity is

$$\frac{mr^2g\theta}{3l} \text{ nearly, for one string;}$$

and

$$\frac{mr^2g\theta}{l} \text{ for all the strings;}$$

$$\therefore\ \frac{d^2\theta}{dt^2} = -\frac{mr^2g\theta}{lmk^2}$$

$$= -\frac{4g\theta}{l}, \text{ as } k^2 = \frac{1}{4}r^2.$$

Hence, integrating,

$$\left(\frac{d\theta}{dt}\right)^2 = C - \frac{8g\theta^2}{2l} = C - \frac{4g\theta^2}{l};$$

or,

$$\left(\frac{d\theta}{dt}\right)^2 = -\frac{4g}{l}(a^2-\theta^2), \text{ supposing } a \text{ to be the greatest}$$

value of θ.

It follows, therefore, that the time of a small oscillation is

$$-\left(\frac{l}{2g}\right)^{\frac{1}{2}}\int_a^{-a}\frac{d\theta}{\sqrt{(a^2-\theta^2)}} = \left(\frac{l}{2g}\right)^{\frac{1}{2}}\pi.$$

[Second Solution. *Mr. Thomas Weddle.*]

Let l = length of each string, of which we shall neglect the weight (this might, however, be easily taken into account, but the result would be somewhat less simple), and a, k and ρ, the radius, thickness and density of the plate. Take the centre of the plate when at rest for the origin of rectangular coordinates, the axis of z being vertical; let $(r\cos\theta, r\sin\theta, 0)$ be the position of any particle $m(=\rho k\, r\, dr\, d\theta)$ when the plate is at rest, and (x, y, z) denote the same at the end of the time t, r then making an angle Ψ with its position of equilibrium. We readily get

$$x = r\cos(\theta+\Psi), \qquad y = r\sin(\theta+\Psi), \qquad z = \frac{a^2}{2l}\Psi^2 \ldots\ldots (1),$$

the last resulting from Ψ being small.

Hence,

$$dx^2+dy^2+dz^2 = (r^2+\frac{a^4}{l^2}\Psi^2)d\Psi^2$$

$$= r^2.\ d\Psi^2, \text{ nearly}\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Moreover, by the principle of the conservation of vis viva, we have

$$\Sigma mv^2 = 2\Sigma.\int m(X dx + Y dy + Z dz).$$

Now, $v^2 = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = (2)\ r^2\frac{d\Psi^2}{dt^2}$, $X=0$, $Y=0$, and $Z=-g$;

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$$\therefore \quad \Sigma mr^2 . \frac{d\Psi^2}{dt^2} = -2g . \Sigma m . \int dz$$
$$= 2g . \Sigma m . (C - z)$$
$$= (1) \frac{a^2 g}{l} . \Sigma m . (\Psi'^2 - \Psi^2) \ldots\ldots\ldots (3),$$

where Ψ' is the initial value of Ψ.

Also $\Sigma m = k\rho\pi a^2$, and $\Sigma mr^2 = k\rho . \int_0^a r^3 dr . \int_0^{2\pi} d\theta = \frac{1}{2} k\rho\pi a^4$. Hence (3) becomes

$$\frac{d\Psi^2}{dt^2} = \frac{2g}{l} (\Psi'^2 - \Psi^2) \ldots\ldots\ldots\ldots\ldots (4);$$

$$\therefore \frac{dt}{d\Psi} = \left\{\frac{l}{2g}\right\}^{\frac{1}{2}} . \frac{-1}{\sqrt{\Psi'^2 - \Psi^2}}.$$

Integrate, recollecting that $t = 0$ when $\Psi = \Psi'$,

$$\therefore \quad t = \left\{\frac{l}{2g}\right\}^{\frac{1}{2}} . \cos^{-1} \frac{\Psi}{\Psi'} \ldots\ldots\ldots\ldots\ldots (5).$$

When $\Psi = -\Psi'$, (5) becomes $\pi \left\{\frac{l}{2g}\right\}^{\frac{1}{2}}$, which is the time of an oscillation; it is the same as that of a simple pendulum whose length $(= \frac{1}{2}l)$ is half that of the string.

A somewhat similar question was proposed in the Lady's and Gentleman's Diary for 1841, solutions to which may be found in the Diary for 1842. See also Walton's Mechanical Problems, p. 333.

The Proposer also gives a good solution.

C. *Mr. Weddle, Newcastle.*

Required the form of the equation to the hyperboloids whose asymptotic cones touch three given planes.

[Solution. *Mr. Weddle, the Proposer.*]

Take the given for coordinate planes. Since the hyperboloids must evidently have the origin for centre, their equations must be of the form

$$A^2x^2 + B^2y^2 + C^2z^2 + 2Dyz + 2Exz + 2Fxy = \pm 1 \ldots\ldots\ldots (1),$$

and it is clear that the equation to the asymptotic cones will be

$$A^2x^2 + B^2y^2 + C^2z^2 + 2Dyz + 2Exz + 2Fxy = 0 \ldots\ldots\ldots (2).$$

The traces of (2) on the planes of yz, xz, and xy are respectively

$$\left.\begin{aligned} B^2y^2 + C^2z^2 + 2Dyz &= 0 \\ A^2x^2 + C^2z^2 + 2Exz &= 0 \\ A^2x^2 + B^2y^2 + 2Fxy &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (3).$$

Now since the cone (2) touches the coordinate planes, each of the equations (3) must be a complete square; $\therefore$ $D = -BC$, $E = -AC$, and $F = \pm AB$. (Since the signs of A, B, C are arbitrary, it is obviously unnecessary to prefix the double sign to the values of any of the quantities D, E, F, except one). Moreover, if the upper sign of the value of F be taken, (2) will be found to be a complete square; hence, rejecting this value, we get

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$$A^2x^2+B^2y^2+C^2z^2-2BCyz-2ACxz-2ABxy=0\ldots.(4)^*$$

as the general equation to the cones touching the coordinate planes. Hence the equation, (1), to the hyperboloids whose asymptotic cones touch the coordinate planes, is

$$A^2x^2+B^2y^2+C^2z^2-2BCyz-2ACxz-2ABxy=\pm 1\ldots.(5),$$

A, B and C being arbitrary. This equation will denote hyperboloids of one or two sheets according as the upper or lower sign is taken. To justify this assertion it is sufficient to observe that the coordinate planes, being tangent to the asymptotic cones, must intersect the former surfaces but not the latter.

If the coordinate planes be chosen so that

$$u=0,\quad v=0,\quad w=0\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

may be equations to the given planes, then x, y, z, continuing to denote the coordinates when the latter are taken as coordinate planes, we shall have $x=au$, $y=\beta v$, and $z=\gamma w$, a, β and γ being constants; substitute these in (5) and we have an equation of the form

$$a^2u^2+b^2v^2+c^2w^2-2bcvw-2acuw-2abuv=\pm 1\ldots\ldots(5'),$$

a, b, and c being now the arbitrary constants. This equation which is of the very same form as (5) is that of the hyperboloids touching the three given planes (6).

* Equation (4) is equivalent to equation (14) given by me at p. 151, "Mathematician," Vol. I.; the above has, however, the advantage of being somewhat more symmetrical.

MATHEMATICAL EXERCISES—*(continued.)*

107. *W. Burns, LL.D., Rochester.*

Given the vertical angle of a triangle, the ratio of the radii of the circles inscribed in the two triangles formed by the line bisecting the vertical angle and meeting the base, and the distance between the centres of these circles, to construct the triangle.

108. *Mr. John Riddle, Nautical School, Greenwich.*

An observer a short distance inland, notes the time when a steamer is just passing a land-mark on the beach, and m minutes afterwards the angle subtended by the vessel and the mark is α, and again in n minutes their angular distance is $\alpha+\beta$. The bearing of the land-mark being known, required the course of the ship?

109. *Mr. R. Harley, Northwich, Cheshire.*

ACB is a semicircle whose diameter is AB and vertex C: at any point D in AB draw a perpendicular DE meeting the semicircle in E and draw CD. On CD take CP such that $CP.CD=DE^2$, and find the quadrature of the curve which is the locus of the point P, without the aid of the differential calculus.

110. *Mr. J. W. Elliott, Greatham, Stockton-on-Tees.*

A straight uniform beam is placed upon two rough planes, whose coefficients of friction are μ and μ', and inclinations to the horizon α and β; to find the limiting value of the angle of inclination of the beam to the horizon at which it will rest, and the relation between the weight of the beam and each of the pressures.

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111. *Mr. G. F. Lacey, Woolwich Common.*

Two perfectly smooth and hard planes, the one horizontal and the other vertical, meet each other; two equal balls whose elasticities are (e, e_1) are projected along the horizontal plane in parallel lines so as to impinge on the vertical in the points E, E_1, then if e_1 be greater than e, their paths after reflexion will intersect in some point D. Shew that if the same balls be projected in parallel lines under any other angle of incidence, the points E, E_1 remaining the same, the locus of the successive intersections D_1, D_2, *etc.* of their paths corresponding to successive angles of incidence θ_1, θ_2, *etc.* .. will be a straight line perpendicular to the vertical plane; and that their distances on this line from the vertical plane will be to each other as $\cot\theta : \cot\theta_1 : \cot\theta_2$, *etc.*.....

112. *Mr. Fenwick.*

Let four points in a plane be so posited that one of them is within the triangle formed by joining the other three. Then having joined the points, taken in pairs, by six straight lines, these lines (produced if necessary) meet in three more points, which lie in an ellipse passing through the middle points of the six lines. Required proof.

113. *Mr. J. F. Heather, B.A., R.M. Academy.*

Two forces applied to a material particle, being resolved into components, which have their lines of direction along two coordinate axes, inclined to one another at an angle θ; it is required to find the position of one of these axes, so that the sum of the components along it shall be constant, whatever be the value of θ.

114. *Mr. Thomas Weddle, Newcastle-upon-Tyne.*

Inscribe a maximum polygon of n sides in a given segment of a parabola.

115. *Mr. W. S. B. Woolhouse, Editor of the Lady's and Gentleman's Diary.*

Two slender and perfectly smooth circular rings are linked together, and being then freely suspended, are made to vibrate first in the plane of the upper, and afterwards in the plane of the lower ring. Compare the times of oscillation.

116. *By J. J.*

There are n candidates for a situation, to whom the voters attribute every degree of merit contained between the limits 0 and m. Shew how to find the mean value of all the candidates' merits.

117. *Mr. Thomas Weddle, Newcastle-upon-Tyne.*

Let a closed curve be described on any surface, (the equation of the latter being $z = f(x, y)$,) and with the origin of coordinates as vertex and this curve as director, let a conical surface be described. If V denote the volume of the solid sector comprised between the two surfaces, and $x_0 y_0 z_0$ the coordinates of its centre of gravity, it is required to show that

$$V = \pm \tfrac{1}{3}\iint(z - px - qy)\,dx\,dy,$$

$$V.x_0 = \pm \tfrac{1}{4}\iint x(z - px - qy)\,dx\,dy, \text{ etc.}$$

where $z = f(x, y)$, and the partial differential coefficients, $p = \dfrac{dz}{dx}$, and $q = \dfrac{dz}{dy}$, are derived from $z = f(x, y)$. The integration must extend over all the elements, $dx\,dy$, of the space contained within the projection of the closed curve on the plane of xy.

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THE MATHEMATICIAN.

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ON THE PLANE QUADRILATERAL.

[*Mr. Fenwick.*]

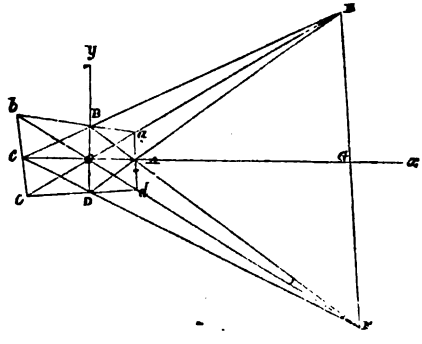
PART I.

Properties of the complete inscribed and circumscribed quadrilaterals, to a system of curves of the second order.

My chief object in view in the first part of the subsequent discussion is, to establish a connected system of lines and points, by which *every* property of the plane quadrilateral may readily be deduced. How far I have succeeded in this respect, must be left to others to determine. At any rate I expect that I have added to the many properties already known of the quadrilateral and the series of conic sections by which it is necessarily circumscribed.

The second part relates to properties of the centres of gravity of a quadrilateral and the triangles of which it is composed. Though many of these are known, their investigation, I trust, will be found interesting to some of our readers. When I say *known*, I mean they are known to those mathematicians who read mathematical works published on the continent, for I am not aware that they are given in any English work.

Produce the opposite sides of the quadrilateral ABCD to meet in E and F, and let O be the intersection of the diagonals AC, BD. Join EO, FO and EF.



Take the diagonals AC, BD, for axes of coordinates, and denote the points A,B,C,D, thus:—

$$\left(\frac{1}{p}, 0\right), \left(0, \frac{1}{m}\right), \left(-\frac{1}{q}, 0\right), \left(0, -\frac{1}{n}\right).$$

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Then if the conic sections which pass through A, B, C, D, be represented by the equation

$$ay^2+2bxy+cx^2+2dy+2ex+1=0 \quad \ldots\ldots\ldots\ldots \quad (1),$$

we shall have at the points $\left(\frac{1}{p},0\right),\left(-\frac{1}{q},0\right)$,

$$\frac{c}{p^2}+\frac{2e}{p}+1=0,\ \frac{c}{q^2}-\frac{2e}{q}+1=0.$$

These give for c and e, the values

$$c=-pq, \quad \text{and } e=-\tfrac{1}{2}(p-q).$$

And at the points $\left(0,\frac{1}{m}\right),\left(0,-\frac{1}{n}\right)$, we have

$$\frac{a}{m^2}+\frac{2d}{m}+1=0,\ \frac{a}{n^2}-\frac{2d}{n}+1=0,$$

which give $\quad a=-mn, \quad \text{and } d=-\tfrac{1}{2}(m-n).$

The equation (1) therefore becomes

$$-mny^2+2bxy-pqx^2-(m-n)y-(p-q)x+1=0 \quad \ldots(1').$$

Since b is arbitrary, this equation represents an indefinite number of ellipses, an indefinite number of hyperbolas, and two parabolas.* If, however, we had taken m, n, or p, q with the *same sign*, (1') would have represented *hyperbolas* only. In the former case, one of the four points is *without* the triangle formed by joining the other three, in the latter it is *within* such triangle. Hence

(*a*) *If one of four points be without the triangle formed by joining the other three, there can pass through the points, an indefinite number of ellipses, an indefinite number of hyperbolas, and two parabolas. If, however, the point be within such triangle, hyperbolas only can pass through the given points.*

Differentiating the equation (1') we get

$$\frac{dy}{dx}=\frac{2pqx-2by+(p-q)}{2bx-2mny-(m-n)}=\frac{y}{x}\cdot\frac{2pqx^2-2bxy+(p-q)x}{2bxy-2mny^2-(m-n)y}.$$

But $2bxy=mny^2+pqx^2+(m-n)y+(p-q)x-1$. Hence the equation of a tangent to (1') at a point (xy) is

$$Y-y=\frac{y}{x}\cdot\frac{pqx^2-mny^2-(m-n)y+1}{pqx^2-mny^2+(p-q)x-1}(X-x) \quad \ldots\ldots(2).$$

Let this tangent pass through the point E, $(x'y')$. Then x, y, being *variable quantities*, the equation

$$y'-y=\frac{y}{x}\cdot\frac{pqx^2-mny^2-(m-n)y+1}{pqx^2-mny^2+(p-q)x-1}(x'-x) \quad \ldots\ldots(3),$$

will obviously contain (1'), and the chord of contact corresponding to the point $(x'y')$. Denoting, therefore, the chord of contact by

$$a_1y+b_1x+c_1=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and putting u for (1'), the equation

$$u(a_1y+b_1x+c_1)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

* That there are *two* parabolas and two only, will be obvious from this consideration, that if u be the value of b which satisfies the equation $b^2-mnpq=0$, $-u$ for b will also satisfy the same equation; but as m, n, p, q are constant, no other value of b can satisfy this equation.

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must be identical with (3). Developing each of these identical equations, and equating the coefficients of the homologous terms of the variables, we have

$$a_1 = x', \quad b_1 = y', \quad c_1 = 0.$$

The chord of contact (4) consequently becomes

$$x'y + y'x = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Again, the equations of AB, BC, CD, DA, are

$$my + px = 1 \ldots\ldots (7), \qquad my - qx = 1. \ldots\ldots\ldots (8),$$
$$-ny - qx = 1 \ldots\ldots (9), \qquad -ny + px = 1 \ldots\ldots\ldots (10).$$

Take (9) from (7) and there results for FO, the equation

$$(m+n)y + (p+q)x = 0 \ldots\ldots\ldots\ldots\ldots (11).$$

Resolving (8) and (10) for x and y, we get for the coordinates of E,

$$x' = \frac{m+n}{mp-nq}, \quad y' = \frac{p+q}{mp-nq} \ldots\ldots\ldots\ldots\ldots (12).$$

Hence (11) becomes by means of (12),

$$x'y + y'x = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6').$$

We see now by the equations (6) and (6'), that the chord of contact corresponding to the point E, $(x'y')$, is the line FO. This line is represented by either of the equations

$$x'y + y'x = 0, \text{ or, } (m+n)y + (p+q)x = 0 \ldots\ldots\ldots (13).$$

In a similar way it is shown that the chord of contact corresponding to the point F, $(x''y'')$, is the line EO, and that this line is represented by either of the equations

$$x''y + y''x = 0, \text{ or, } (m+n)y - (p+q)x = 0 \ldots\ldots (14),$$

the coordinates of E being

$$x'' = \frac{m+n}{np-mq}, \; y'' = \frac{p+q}{np-mq} \ldots\ldots\ldots\ldots\ldots (15).$$

Hence this property:—

(*b*) *If a series of conic sections be circumscribed to a given quadrilateral figure, whose diagonals intersect in* O, *and from* E *and* F, *the intersections of the two pairs of opposite sides, tangents be drawn, then will all the points of contact corresponding to the point* E, *lie in* OF, *and those corresponding to* F, *will lie in* OE.*

We will now discuss the properties of any circumscribed quadrilateral, *a b c d*.

Tangents at the points A, C, B, D, in reference to (1') are,

$$\{2b - p(m-n)\}y - p(p+q)x + (p+q) = 0 \ldots\ldots (16),$$
$$\{2b + q(m-n)\}y - q(p+q)x - (p+q) = 0 \ldots\ldots (17),$$
$$m(m+n)y - \{2b - m(p-q)\}x - (m+n) = 0 \ldots\ldots (18),$$
$$n(m+n)y - \{2b + n(p-q)\}x + (m+n) = 0 \ldots\ldots (19).$$

* Demonstrations on other principles, chiefly by poles and polars, have been given of this and some other properties in this paper, by different writers. They are given here as *parts* of the "connected system of lines and points," which it is my object to establish.

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Combine (16) and (18), and also (17) and (19), so that the absolute terms in the resulting equations may be zero. Then we have for the lines aO, cO, the equations

$$(m+n)y-(p+q)x=0 \ldots(20), \quad (m+n)y-(p+q)x=0\ldots.(21).$$

As these are identical, and as both pass through the origin, the line which joins the points a and c, passes through O, the intersection of the diagonals. Also, comparing the equations (14), (20) and (21), we see that the line ac passes through the point E.

We might shew in a similar way that the line bd passes through the points O and F.

Again, combine (16) and (17) by subtraction, and we have

$$(m-n)y+(p-q)x=2\ldots\ldots\ldots\ldots\ldots(22).$$

This is the equation of a line passing through the intersection of (16) and (17), or of ad and bc. It is in fact the equation of the line EF. For combining (7) and (9), and also (8) and (10), by addition, we have the equations

$$(m-n)y+(p-q)x=2\ldots(23), \quad (m-n)y+(p-q)x=2\ldots(24).$$

Now (23) passes through the intersection of (7) and (9), or through the point F; and (24) through the intersection of (8) and (10), or the point E. It follows, therefore, that (23) or (24) is the equation of EF. But each of these is identical with (22), and hence (22) is the equation of EF.

Consequently, the lines ad and bc, whatever be the value of the arbitrary b, intersect in the line EF. Similarly, the lines ab, cd, intersect in EF. Hence the following properties.

If in any number of curves of the second order passing through the same four points A, B, C, D, we inscribe a quadrilateral ABCD, and produce the opposite sides to meet in E and F, and describe quadrilaterals about the curves whose sides touch the same at the angular points of the inscribed quadrilateral, and produce the opposite sides of the circumscribed quadrilaterals till they meet: then

(*c*) *The diagonals of all the circumscribed quadrilaterals will pass through the same point, the intersection of the diagonals of the inscribed one.*

(*d*) *The points of intersection of the opposite sides of the circumscribed figures will lie in the line* EF.

Again, in the equation of EF, (22), let $y=0$, then $x=\mathrm{OG}=\dfrac{2}{p-q}$.

Hence $\mathrm{AG}=\dfrac{2}{p-q}-\dfrac{1}{p}=\dfrac{p+q}{p(p-q)}$, and $\mathrm{CG}=\dfrac{2}{p-q}+\dfrac{1}{q}=\dfrac{p+q}{q(p-q)}$.

Consequently, $\mathrm{AG}:\mathrm{CG}::\dfrac{1}{p}:\dfrac{1}{q}::\mathrm{OA}:\mathrm{OC}$, or the line AC is divided harmonically at the points A, C, O, G.

In a similar way it is shewn that BD is divided harmonically at the points B, D, O, and the intersection of BD with EF.

And, moreover, by a process exactly analogous it may be proved, that EBC,

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EAD are divided similarly by FO; and FAB, FDC by EO. Hence this theorem:—

(*e*) *If we join four given points* A, B, C, D, *taken in pairs, by six straight lines* AB, BC, CD, DA, AC, BD, *so that* AB, CD *may meet in* F, BC, AD *in* E, *and* AC, BD *in* O; *and also join* EO, FO, EF: *then with respect to each of the six lines (produced if necessary), there will be four points of intersection, at which each of the lines will be harmonically divided.*

This property in connection with any of the conic sections is this:

(*f*) *If from the intersection of two tangents to a conic section, a line be drawn to meet the curve and the chord of contact, this line will be divided harmonically at the four points.*

Let us advert again to equation (1′). By partial differentiation that equation gives the two equations,

$$-2mny+2bx-(m-n)=0,$$
$$-2pqx+2by-(p-q)=0.$$

Eliminate the arbitrary b from these and we have

$$2mny^2-2pqx^2+(m-n)y-(p-q)x=0 \ldots\ldots (25).$$

This is the locus of the centres of all conic sections which pass through A, B, C, D. Hence, when (as we have assumed) m, n and p, q have *different signs*, the locus of the centres of (1′) is a *hyperbola*. If we had taken m, n, or p, q with the *same sign*, then (25) would have been an *ellipse*. Whence

(*g*) *If one of four points be without the triangle formed by joining the other three, the locus of the centres of all conic sections which pass through the given points is a hyperbola: if, however, that point be within such triangle, the locus is an ellipse.**

As the coefficient of xy in (25) is zero, the axes, or the diagonals AC, BD, are parallel to conjugate diameters. Had we taken EC, ED, or FC, FB as axes, the equation of the locus would have been of the same form. Hence

(*h*) *The curve* (25) *has three pairs of conjugate diameters parallel, respectively, to the sides* AB, CD; BC, AD; AC, BD.

Again, the coordinates of the middle points of the opposite sides AB, CD, are

$$\left(\frac{1}{2p}, \frac{1}{2m}\right), \left(-\frac{1}{2q}, -\frac{1}{2n}\right) \ldots\ldots\ldots\ldots (26),$$

of the middle points of BC, AD,

$$\left(-\frac{1}{2q}, \frac{1}{2m}\right), \left(\frac{1}{2p}, -\frac{1}{2n}\right) \ldots\ldots\ldots\ldots (27),$$

and of the middle points of the diagonals,

$$\left(\frac{q-p}{2pq}, 0\right), \left(0, \frac{n-m}{2mn}\right) \ldots\ldots\ldots\ldots (28).$$

* Investigations of this locus have already appeared in the Mathematician, but this discussion, I hope, will be considered better than any that has been given.

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Now these coordinates (26, 27, 28) satisfy, respectively, the equation (25), as do those of E and F, given in (12) and (15); and as (25) passes through the origin, or the intersection of the diagonals AC, BD, we have this property:—

(*k*) *If four given points, taken in pairs, be joined by six straight lines, the intersections of the three pairs of opposite sides (the diagonals being one pair), and the middle points of the six lines, lie in the conic section which is the locus of the centres of all conic sections which pass through the four points.*

By (*g*) and (*k*) we deduce the following:—

(*l*) *If four points, taken in pairs, be joined by six straight lines, the intersections of the three pairs of opposite lines, and the middle points of the six lines, lie in an ellipse or hyperbola, according as one of the four points is within or without the triangle formed by joining the other three.*

The partial differential of (25) gives

$$4mny+(m-n)=0,$$
$$4pqx+(p-q)=0.$$

The coordinates, therefore, of the centre of (25), are

$$x=\frac{q-p}{4pq},\quad y=\frac{n-m}{4mn}\ldots\ldots\ldots\ldots\ldots\ldots(29).$$

Also, the equations of the lines through the three pairs of points, (26), (27), and (28), or the equations of the three straight lines which join the middle points of the opposite sides and of both diagonals, are

$$2mn(p+q)y-2pq(m+n)x=np-mq\ldots\ldots\ldots(30),$$
$$2mn(p+q)y+2pq(m+n)x=nq-mp\ldots\ldots\ldots(31),$$
$$-\frac{2mn}{m-n}y-\frac{2pq}{p-q}x=1\ldots\ldots\ldots\ldots\ldots(32).$$

Now (30) and (31) intersect in a point whose coordinates are

$$x=\frac{q-p}{4pq},\ y=\frac{n-m}{4mn}\ldots\ldots\ldots\ldots\ldots\ldots(33),$$

which satisfy also (32); hence by (29),

(*m*) *The three straight lines which join the middle points of the opposite sides and of both diagonals of a quadrilateral, intersect in the centre of the curve which is the locus of the centres of all conic sections which pass through the angular points of the quadrilateral.*

We will next give the equations of all curves of the second order which circumscribe, respectively, the triangles ADF, BCF, EAB, ECD. For this purpose it is advantageous to employ a pair of the opposite sides AB, CD, or AD, BC, as axes. Hence, putting $FA=g$, $FD=h$, $FB=k$, $FC=l$, and taking FC, FB, for axes, the equations of the sides AB, BC, CD, DA, will be

$$x=0,\ u=ky+lx-kl=0,\ y=0,\ v=gy+hx-gh=0.$$

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Consequently, λ_1, λ_2, μ_1, μ_2, *etc.*, being arbitrary constants, all conic sections which circumscribe, respectively, ADF, BCF, EAB, ECD, are

$$xy + \lambda_1 xv + \mu_1 yv = 0 \ldots \ldots \ldots \ldots \ldots (34),$$
$$xy + \lambda_2 xu + \mu_2 yu = 0 \ldots \ldots \ldots \ldots \ldots (35),$$
$$uv + \lambda_3 xu + \mu_3 xv = 0 \ldots \ldots \ldots \ldots \ldots (36),$$
$$uv + \lambda_4 yu + \mu_4 yv = 0 \ldots \ldots \ldots \ldots \ldots (37).$$

These will be *circles* when the coefficients of y^2 and x^2 are equal, and the coefficient of xy divided by that of y^2 or x^2 is equal to $2\cos\phi$; ϕ being the angle contained by the axes. These conditions give for the arbitrary quantities, the following values :—

$$\lambda_1 = -\frac{g}{g^2+h^2-2gh\cos\phi}, \quad \mu_1 = -\frac{h}{g^2+h^2-2gh\cos\phi},$$
$$\lambda_2 = -\frac{k}{k^2+l^2-2kl\cos\phi}, \quad \mu_2 = -\frac{l}{k^2+l^2-2kl\cos\phi},$$
$$\lambda_3 = \frac{k(g^2+h^2-2gh\cos\phi)}{gl-hk}, \quad \mu_3 = -\frac{g(k^2+l^2-2kl\cos\phi)}{gl-hk},$$
$$\lambda_4 = -\frac{l(g^2+h^2-2gh\cos\phi)}{gl-hk}, \quad \mu_4 = \frac{h(k^2+l^2-2kl\cos\phi)}{gl-hk}.$$

Substituting the values of λ_1, λ_2, *etc.*, in (34) and (35) and adding the resulting equations, we have

$$y^2+2\cos\phi.xy+x^2-\tfrac{1}{2}(h+l)y-\tfrac{1}{2}(g+k)x=0 \ldots \ldots \ldots (38).$$

This is the equation of a circle. It evidently passes through the points common to (34) and (35).

Moreover, the equation which results from adding (36) and (37), after substituting the values of λ_3, λ_4, *etc.*, is satisfied by the equations,

$$y^2+2\cos\phi.xy+x^2-\tfrac{1}{2}(h+l)y-\tfrac{1}{2}(g+k)x=0 \ldots \ldots \ldots (39),$$
$$\{(k-g)(hl+gk)-2gk(l-h)\cos\phi\}y-\{(l-h)(hl+gk)-2hl(k-g)\cos\phi\}x + (hl+gk)(gl-hk)=0 \ldots \ldots (40).$$

Hence it will be obvious by (38), (39), (40), that when (34 35, 36, 37) are circles, they intersect in a point on the line (40). Whence this theorem,

(*n*) *If we produce the pairs of opposite sides* AB, CD, and AD, BC, *of a quadrilateral* ABCD, *till they intersect, and describe a circle about every one of the four triangles so formed, the four circles will intersect in the same point.*

These circles have some other curious relations, but as I have exceeded the limits to a discussion of this kind, I will merely give the enunciations of some additional properties, and leave their investigation as an exercise for the student.

(*o*) *The centres of the four circles lie in another circle which passes through* P, *the point common to all.*

(*p*) *Perpendiculars from* P *on the sides* AB, BC, CD, DA, *intersect these sides in four points which range on the same straight line.*

(*q*) *When the four points* A, B, C, D, *lie in the circumference of a circle, the difference of the squares of the radii of the first two circles is equal to the difference of the squares of the radii of the other two.*

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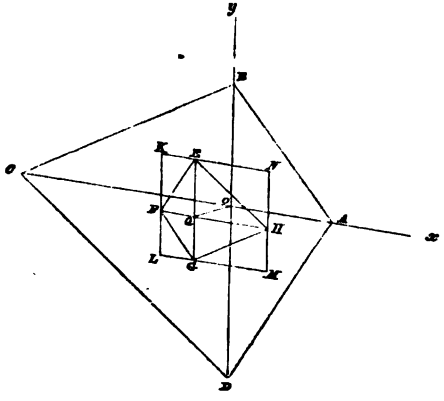
PART II.

Properties of the centres of gravity of a quadrilateral and the triangles of which it is composed.

Let ABCD be a trapezium or quadrilateral, the diagonals of which intersect in O. Let E, F, G, H, be the centres of gravity of the triangles ABC, BCD, ADC, BAD; and K, L, M, N, those of the four triangles which have the common vertex O. Then taking the two diagonals AC, BD for axes of coordinates, and putting



$OA = a$, $OB = b$, $OC = c$, $OD = d$,

the coordinates of the four points A, B, C, D, are

$(a, 0), (0, b), (-c, 0), (0, -d)$ ....(41).

Moreover, the coordinates of the middle points of the diagonals AC, BD, are

$\{\frac{1}{2}(a-c), 0\}$, and $\{0, \frac{1}{2}(b-d)\}$.....(42).

Hence, as the points E and G are the centres of gravity of the triangles ABC, ADC, the coordinates of these points (assuming the well known property of the centre of gravity of a triangle), are

$\{\frac{1}{3}(a-c), \frac{1}{3}b\}$, and $\{\frac{1}{3}(a-c), -\frac{1}{3}d\}$ .....(43).

Similarly, the coordinates of F and H, are

$\{-\frac{1}{3}c, \frac{1}{3}(b-d)\}$, and $\{\frac{1}{3}a, \frac{1}{3}(b-d)\}$.....(44).

Consequently, the equations of the lines EG, FH, are

$x = \frac{1}{3}(a-c)$.....(45), $y = \frac{1}{3}(b-d)$.....(46).

Now it will be obvious that the centre of gravity of the whole figure ABCD lies in the line EG, and also in the line FH.

It hence follows that the intersection of (45) and (46), or the point whose coordinates are

$x = \frac{1}{3}(a-c)$, $y = \frac{1}{3}(b-d)$.....(47),

is the centre of gravity of the whole quadrilateral. The equations (45), (46) and (47) suggest the following methods of finding the centre of gravity of a quadrilateral or trapezium ABCD, whose diagonals AC, BD intersect in O:—

(*r*) *Draw the diagonals* AC, BD, *and find the centres of gravity* E, F, *of the triangles* ABC, BCD: *draw* EG, FH, *parallel, respectively, to* BD, AC; *then is their intersection* Q, *the centre of gravity required.*

(*s*) *Draw the diagonals* AC, BD. *Set off upon* OC, *the greater of the segments* OC, OA, *a part equal to one third of the difference of the segments* OC, OA, *also upon* OD *a part equal to one third of the difference of the segments* OD, OB; *and with these as sides complete*

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the parallelogram OQ: *then is* Q *the centre of gravity of the quadrilateral.*

The second method (*s*) is well known. It first appeared, I believe, in the *Mechanic's Magazine*, and subsequently in the *Lady's Diary for* 1830.

Again, the absolute length of EQ by (43) and (44) is

$$= \tfrac{1}{3}b + \tfrac{1}{3}(d - b) = \tfrac{1}{3}d = \tfrac{1}{3}\text{OD}.$$

We have therefore another elegant method of finding the centre of gravity of a quadrilateral.

(*t*) *Draw the diagonals* AC, BD, *of the quadrilateral: find the centre of gravity* E, *of the triangle* ABC; *then if we draw* EG *parallel to* BD *and make* EQ *one third of the segment* OD, Q *will be the centre of gravity of the quadrilateral.*

Again, the coordinates of the middle points of the opposite sides CB, AD, are

$$(-\tfrac{1}{2}c, \tfrac{1}{2}b), \text{ and } (\tfrac{1}{2}a, -\tfrac{1}{2}d) \ldots\ldots\ldots\ldots\ldots\ldots (48),$$

and of the middle points of AB, CD,

$$(\tfrac{1}{2}a, \tfrac{1}{2}b), \text{ and } (-\tfrac{1}{2}c, -\tfrac{1}{2}d) \ldots\ldots\ldots\ldots\ldots\ldots (49).$$

Hence, lines through the points (48) and (49), are

$$y - \tfrac{1}{2}b = -\frac{b+d}{a+c}(x + \tfrac{1}{2}c) \ldots\ldots\ldots\ldots\ldots\ldots (50),$$

$$y - \tfrac{1}{2}b = \frac{b+d}{a+c}(x - \tfrac{1}{2}a) \ldots\ldots\ldots\ldots\ldots\ldots (51).$$

These intersect in a point whose coordinates are

$$x = \tfrac{1}{4}(a - c),\ y = \tfrac{1}{4}(b - d) \ldots\ldots\ldots\ldots\ldots\ldots (52).$$

It therefore follows from (47) and (52) that the two pairs of lines which bisect the opposite sides of the quadrilateral, intersect in a point P, on OQ, and that PQ is one third of OP. Hence this additional method of finding the centre of gravity of a quadrilateral:—

(*u*) *Bisect the two pairs of opposite sides by two lines which meet in* P,* *and join* P *and the intersection* O, *of the diagonals. Produce* OP *till the produced part* PQ *is one third of* OP; *then is* Q *the centre of gravity required.*

Another method, of considerable elegance, said to be due to CARNOT, is noticed by one of our correspondents at p. 268, of our last number.

We will now discuss some properties of the centres of gravity of the triangles ABC, BCD, CDA, BAD.

The equations of the four lines EF, FG, GH, HE (the coordinates of the points E, F, G, H, are already given) are

$$y - \tfrac{1}{3}b = \frac{d}{a}\left\{x - \tfrac{1}{3}(a - c)\right\} \ldots\ldots\ldots\ldots\ldots\ldots (53),$$

$$y + \tfrac{1}{3}d = -\frac{b}{a}\left\{x - \tfrac{1}{3}(a - c)\right\} \ldots\ldots\ldots\ldots\ldots\ldots (54),$$

$$y + \tfrac{1}{3}d = \frac{b}{c}\left\{x - \tfrac{1}{3}(a - c)\right\} \ldots\ldots\ldots\ldots\ldots\ldots (55),$$

$$y - \tfrac{1}{3}b = -\frac{d}{c}\left\{x - \tfrac{1}{3}(a - c)\right\} \ldots\ldots\ldots\ldots\ldots\ldots (56).$$

* The point P is not given in the diagram, but its position is easily conceived.

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Also, the equations of AB, BC, CD, DA, are

$$y = -\frac{b}{a}x + b \quad \ldots\ldots(57),$$

$$y = \frac{b}{c}x + b \quad \ldots\ldots(58),$$

$$y = -\frac{d}{c}x - d \quad \ldots\ldots(59),$$

$$y = \frac{d}{a}x - d \quad \ldots\ldots(60).$$

Hence, the four pairs of lines, (53, 60), (54, 57), (55, 58), and (56, 59), are respectively parallel; that is, the four pairs of lines, EF, DA; FG, AB; GH, BC; EH, CD; are respectively parallel. Whence the quadrilateral EFGH is similar to ABCD, but placed in a reversed position. We have shewn, moreover, that EQ is one third of DO. In a similar way it is shewn that

$$GQ = \tfrac{1}{3}BO,\ HQ = \tfrac{1}{3}CO,\ FQ = \tfrac{1}{3}AO.$$

Consequently, $EG = \frac{1}{3}BD$, $FH = \frac{1}{3}AC$.

Hence the following properties :—

(*v*) *If we join the centres of gravity of the four triangles which two and two compose a quadrilateral, we shall form another quadrilateral, similar to the former, but reversed in position, and whose area is equal to one ninth of the original quadrilateral.*

(*w*) *Also, the diagonals of the new quadrilateral which intersect in the centre of gravity of the original one, are parallel to the diagonals of the same and are equal respectively to one third of them.*

Let us now consider the centres of gravity of the four triangles formed by the two diagonals, and which have for their common vertex, the point of intersection of the diagonals.

The coordinates of the middle points of OC, OB, OD, OA, being

$$(-\tfrac{1}{2}c, 0),\quad (0, \tfrac{1}{2}b),\quad (0, -\tfrac{1}{2}d),\quad (\tfrac{1}{2}a, 0);$$

the coordinates of the points K, L, M, N, the centres of gravity of the triangles BOC, DOC, AOD, AOB, will be

$$(-\tfrac{1}{3}c, \tfrac{1}{3}b),\quad (-\tfrac{1}{3}c, -\tfrac{1}{3}d),\quad (\tfrac{1}{3}a, -\tfrac{1}{3}d),\quad (\tfrac{1}{3}a, \tfrac{1}{3}b) \quad \ldots\ldots(61).$$

Hence, the equations of the lines KL, LM, MN, KN, are

$$x = -\tfrac{1}{3}c \quad \ldots\ldots(62), \qquad y = -\tfrac{1}{3}d \quad \ldots\ldots(63),$$

$$x = \tfrac{1}{3}a \quad \ldots\ldots(64), \qquad y = \tfrac{1}{3}b \quad \ldots\ldots(65).$$

The figure KLMN is therefore a parallelogram. Its sides are evidently parallel to the diagonals AC, BD, and are equal to EG, FH.

It will moreover be obvious from (43) and (44), that the points E, F, G, H are on the lines (65), (62), (63), and (64), or upon the lines NK, KL, LM, MN.

Again, the equations of lines through K, M, and L, N, respectively, are

$$3(a+c)y + 3(b+d)x = ab - cd \quad \ldots\ldots(66),$$

$$3(a+c)y - 3(b+d)x = bc - ad \quad \ldots\ldots(67).$$

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Combine these so that the resulting line may pass through the origin, and we have the equation

$$(a-c)y-(b-d)x=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (68).$$

This line evidently passes through the point Q, whose coordinates are

$$x=\tfrac{1}{3}(a-c), \quad y=\tfrac{1}{3}(b-d).$$

The lines KM, NL, therefore, intersect in the line OQ. Hence the following theorems:—

(x) *If we join the centres of gravity of the four triangles formed by the two diagonals of a quadrilateral, and which have for their common vertex, the point of intersection of the diagonals, we shall form a parallelogram, whose sides are parallel to the diagonals of the quadrilateral.*

(y) *The sides of this parallelogram contain the centres of gravity of the triangles which two and two compose the quadrilateral, and its diagonals intersect on the line which passes through the centre of gravity and point of intersection of the diagonals, of the quadrilateral.*

(z) *The surface of this parallelogram is two ninths of the surface of the quadrilateral.*

By means of the centres of gravity of the eight triangles which are formed by the diagonals of the original quadrilateral, we can thus form a parallelogram and a new inscribed quadrilateral; and by continuing the process we get a series of parallelograms and inscribed quadrilaterals, the areas of the latter being denoted by the series

$$\frac{1}{9}\,A, \quad \frac{1}{81}\,A, \quad \frac{1}{729}\,A, \textit{ etc.},$$

where A is the area of the original quadrilateral. The diagonals of all the quadrilaterals and parallelograms intersect on the line which passes through the centre of gravity and intersection of the diagonals, of the original quadrilateral.

Moreover, if we join the middle points of the adjacent sides of the original quadrilateral, a parallelogram will be formed, having its sides parallel to the diagonals of the quadrilateral. And if we join the points in which the sides of this parallelogram cut the diagonals of the original quadrilateral, another quadrilateral will be formed similar to the original one; and so on. These parallelograms and quadrilaterals have also a remarkable relation to each other. But we need do no more than direct the student's attention to them. We may remark, however, that all the diagonals of these parallelograms and quadrilaterals intersect, as in the former series, in the line OQ.

Royal Military Academy, Jan. 22nd, 1847.

Note by the Editors, relative to their Elementary Propositions in the Geometry of Coordinates.

We may take this opportunity to state, in reply to several inquiries, that many of our papers in the "Mathematician" may be considered as *continuations* of the "Geometry of Coordinates." When the first part appeared, the publication of the Mathematician was not contemplated.

Our little treatise on this subject has been translated into German by Dr. Wiegand, Professor of Mathematics and Natural Philosophy, at Halle. The following is a translation of Dr. Wiegand's preface.

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"When I first saw the Elementary Propositions, &c., by W. Rutherford and S. Fenwick, of the Royal Military Academy at Woolwich, I was so much struck with the easy and intelligible treatment of the theorems in this branch of geometry, that they seemed to me well deserving to be more generally made known. My expectations have not been disappointed, for the translation of the first number has gone through two editions in a very short time.

"Though the mathematical literature of Germany is by no means poor in comprehensive and excellent works in this particular branch, and indeed we may confidently say, that the works of our "Plucker" are scarcely equalled by all the foreign works put together, yet the important results at which he has arrived are still very far from being known to the mathematical world, in proportion to their merit.

"My object will be fully answered if, by publishing the following papers, I contribute in some measure to the circulation of new views in analytical investigation. Having established a direct correspondence with the authors, the first of whom is already well known to mathematicians in Germany, I shall be able to publish the subsequent numbers immediately after their appearance in England."

ALGEBRAIC DEMONSTRATION OF THE THEOREM

$$1.2.....n = n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n .....etc.$$

[*Mr. J. F. Heather, B. A., R. M. Academy.*]

HIND, in his Algebra, assumes the formula $1.2.3 .....n = n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - etc.$, to be true, for the purpose of proving Sir J. Wilson's theorem for the detection of prime numbers, stating that it may be demonstrated by the higher branches of analysis, but cannot easily be deduced from the principles of common algebra. The following algebraical demonstration may therefore prove acceptable to the readers of the Mathematician.

Let $n^r - \frac{n}{1}(n-1)^r + \frac{n(n-1)}{1.2}(n-2)^r - etc.$ be represented by $f(n)^r$.

Then $f(1)^r = 1$ whatever be the value of r,

and $f(n)^r = n(1-1)^{n-1} = 0$, when n has any value > 1,

$= 1$, —— $n = 1$,

$= \infty$ —— n has any value < 1.

Now $f(n)^r = n\{f(n)^{r-1} + f(n-1)^{r-1}\}$,

$f(n-1)^{r-1} = (n-1)\{f(n-1)^{r-2} + f(n-2)^{r-2}\}$,

$etc. = etc.$

$f(n-r+2)^2 = (n-r+2)\{f(n-r+2)^1 + f(n-r+1)^1\}$;

$\therefore f(n)^r = nf(n)^{r-1} + n(n-1)f(n-1)^{r-2} + etc. + n(n-1)...(n-r+2)f(n-r+2)^1 + n(n-1)...(n-r+2)f(n-r+1)^1$.

By continually decomposing in the same manner the several terms of this series, we shall at length obtain

$$f(n)^r = Af(n)^1 + Bf(n-1)^1 + etc. + \Psi f(n-r+2)^1 + n(n-1)...(n-r+2)f(n-r+1)^1.$$

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If then $r \lessdot n$, n and r being positive, $n \gtrdot 1$, $n, n-1, \ldots n-r+1$, are all $\gtrdot 1$, and consequently $f(n)^1 f(n-1)^1 \ldots f(n-r+1)^1$ are all $= 0$.

If $r \gtrdot n$, $(n-r+1)$ is $\lessdot 1$, and $\therefore f(n-r+1)^1 = \infty$.

But if $r = n$, then $n-r+1=1$, and $(n-r+2), \ldots (n-1)\, n$ are all $\gtrdot 1$, and consequently $f(n)^1 = 0, f(n-1)^1 = 0, \ldots . f(n-r+2)^1 = 0$, and $f(n-r+1)^1 = 1$.

Then
$$\begin{aligned} f(n)^r &= 0, \text{ when } n \gtrdot r, \\ &= \infty \text{ —— } n \lessdot r, \\ &= n(n-1)\ldots(n-r+2) \text{ when } n = r. \end{aligned}$$

Hence $f(n)^n = 1.2.3\ldots\ldots n$, as was to be proved.

Since
$$\begin{aligned} f(n)^0 &= 1 - n\left(1 - \frac{n-1}{2} + etc.\right) = 1, \text{ when } n = 0, \\ &= (1-1)^n = 0, \text{ when } n \text{ is positive}, \\ &= \infty \text{ —— } n \text{ is negative}, \end{aligned}$$

we should have arrived at the same result by carrying the process one step further and obtaining

$$f(n)^r = A_1 f(n)^0 + B_1 f(n-1)^0 + etc. + \Psi f(n-r+1)^0 + n(n-1)\ldots .. (n-r+1) f(n-r)^0 .$$

If $f(n_d)^r$ be taken to represent

$$n^r - \frac{n}{1}(n-d)^r + \frac{n.(n-1)}{1.2}(n-2d)^r - etc,$$

we shall in a similar manner obtain

$$\begin{aligned} f(n_d)^r &= n f(n_d)^{r-1} + n(n-d)\, df(n-d_d)^{r-2} \\ &\quad + n(n-1)(n-2d)d^2 f(n-2d_d)^{r-3} + etc. \\ &\quad + n(n-1)\ldots(n-r+2)(n-r-1d)d^{r-1} f\{n-(r-1)d_d\}^0 \\ &\quad + n(n-1)\ldots(n-r+1).d^r f(n-rd)^0 \\ &= A f(n_d)^0 + B f(n-d_d)^0 + etc. + \Psi f\{n-r-1d_d\}^0 + n(n-1)\ldots \\ &\quad \ldots(n-r+1)d^r f(n-rd)^0 ; \end{aligned}$$

and
$$\begin{aligned} f(n_d)^r &= 0 \text{ when } n \gtrdot rd, \\ &= \infty \text{ when } n \lessdot rd, \\ &= n(n-1)\ldots(n-r+1)d^r \text{ when } n = rd. \end{aligned}$$

Another algebraic demonstration of this theorem is given in the *Cambridge Mathematical Journal*, vol. I., p. 239.

ON CERTAIN DEFINITE INTEGRALS EXPRESSIBLE BY MEANS OF ELLIPTIC FUNCTIONS.

[*The Rev. Brice Bronwin.*]

The various transformations of Elliptic Functions afford easy means of obtaining certain definite integrals, a few of which it is the object of this paper to deduce.

Let $\Delta(k,x) = \sqrt{(1-x^2)(1-k^2x^2)}$, $\Delta(c,y) = \sqrt{(1-y^2)(1-c^2y^2)}$,

$$K = \int_0^1 \frac{dx}{\Delta(k,x)}, \quad K' = \int_0^1 \frac{dx}{\Delta(k',x)} \quad k^2 + k'^2 = 1, \text{ and } k = \frac{2\sqrt{c}}{1+c}.$$

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From Legendre's Exercises, vol. 1, page 68, or Leybourn's Mathematical Repository, vol. III, part 3, page 29, by adopting the symbols given above, we have

$$K=\frac{2K'}{\pi}\log\frac{4}{k'}-\frac{1^2}{2^2}k'^2-\left(1+\frac{2}{3.4}\right)\frac{1^2.3^2}{2^2.4^2}k'^4$$
$$-\left(1+\frac{2}{3.4}+\frac{2}{5.6}\right)\frac{1^2.3^2.5^2}{2^2.4^2.6^2}k'^6-etc..$$

Or, making k and k', and therefore also K and K', change places;

$$\frac{\pi}{2}K'=K\log\frac{4}{k}-\frac{\pi}{2}\left\{\frac{1^2}{2^2}k^2+\left(1+\frac{2}{3.4}\right)\frac{1^2.3^2}{2^2.4^2}k^4+etc.\right\}$$

$$\text{Or } \frac{\pi}{2}K'+K\log k=K\log 4-\frac{\pi}{2}\left\{\frac{1^2}{2^2}k^2+\left(1+\frac{2}{3.4}\right)\frac{1^2.3^2}{2^2.4^2}k^4+etc.\right\}..(a)$$

$$\text{where}\qquad K=\frac{\pi}{2}\left\{1+\frac{1^2}{2^2}k^2+\frac{1^2.3^2}{2^2.4^2}k^4+etc.\right\}$$

Multiplying $(1-k^2x^2)^{-\frac{1}{2}}=1+\frac{1}{2}k^2x^2+\frac{1.3}{2.4}k^4x^4+\frac{1.3.5}{2.4.6}k^6x^6+etc.$,

by $\frac{dx\log x}{\sqrt{1-x^2}}$ and integrating, we have

$$\int_0^1\frac{dx}{\Delta(k,x)}\log x=\Sigma\frac{1.3.5....(2n-1)}{2.4.6...(2n)}k^{2n}\int_0^1\frac{x^{2n}dx}{\sqrt{1-x^2}}\log x.$$

But $d\{(1-x^2)^{\frac{1}{2}}x^{2n-1}\log x\}=(2n-1)\frac{x^{2n-2}dx}{\sqrt{1-x^2}}\log x-2n\frac{x^{2n}dx}{\sqrt{1-x^2}}\log x$
$+x^{2n-2}dx\sqrt{1-x^2}$. The integral of the first member vanishes at both limits;

therefore $\int_0^1\frac{x^{2n}dx}{\sqrt{1-x^2}}\log x=\frac{2n-1}{2n}\int_0^1\frac{x^{2n-2}dx}{\sqrt{1-x^2}}\log x+\frac{1}{2n}\times$

$$\int_0^1 x^{2n-2}dx\sqrt{1-x^2}=\frac{2n-1}{2n}\int_0^1\frac{x^{2n-2}dx}{\sqrt{1-x^2}}\log x+\frac{\pi}{8n}\frac{1.3.5....(2n-3)}{4.6.8....(2n)}.$$

Changing n into $n-1, n-2$, *etc.*, we find $\int_0^1\frac{x^{2n}dx}{\sqrt{1-x^2}}\log x=\frac{1.3..(2n-1)}{2.4....(2n)}\times$

$$\left\{\int_0^1\frac{dx}{\sqrt{1-x^2}}\log x+\frac{\pi}{2}\left(\frac{1}{2n(2n-1)}+\frac{1}{(2n-2)(2n-3)}....+\frac{1}{2.1}\right)\right\}.$$

But by Gregory's Examples, page 485,

$$\int_0^1\frac{dx}{\sqrt{1-x^2}}\log x=-\frac{\pi}{2}\log 2=-\frac{\pi}{4}\log 4.$$

Therefore the general term of the integral is

$$\frac{\pi}{4}\frac{1^2.3^2....(2n-1)^2}{2^2.4^2.....(2n)^2}\left\{-\log 4+1+\frac{2}{3.4}+\frac{2}{5.6}..+\frac{2}{(2n-1)2n}\right\}k^{2n},$$

which, if we put for K its value in the second member of (a), and change the sign of all the terms, we shall find to be the half of the general term of that member. Therefore,

$$\int_0^1\frac{dx}{\Delta(k,x)}\log x=-\frac{\pi}{4}K'-\frac{K}{2}\log k\ .....(1).$$

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Make $x=\left(\frac{1+c}{2}\right)^{\frac{1}{2}}\left(\frac{1+y}{1+cy}\right)^{\frac{1}{2}}$; then $\sqrt{1-x^2}=\left(\frac{1-c}{2}\right)^{\frac{1}{2}}\left(\frac{1-y}{1+cy}\right)^{\frac{1}{2}}$;

$\sqrt{1-k^2x^2}=\left(\frac{1-c}{1+c}\right)^{\frac{1}{2}}\left(\frac{1-cy}{1+cy}\right)^{\frac{1}{2}}$; and we shall find $\frac{dx}{\Delta(k,x)}=\frac{1+c}{2}\frac{dy}{\Delta(c,y)}$.

Multiply this by $\log(1-k^2x^2)=\log\left(\frac{1-c}{1+c}\right)+\log\left(\frac{1-cy}{1+cy}\right)$, and integrate.

Thus $\int_0^1\frac{dx}{\Delta(k,x)}\log\left(1-k^2x^2\right)=\frac{1+c}{2}\int_{-1}^{+1}\frac{dy}{\Delta(c,y)}\log\left(\frac{1-c}{1+c}\right)+$

$$\frac{1+c}{2}\int_{-1}^{+1}\frac{dy}{\Delta(c,y)}\log\left(\frac{1-cy}{1+cy}\right).$$

The last integral of the second member is nothing, the positive and negative elements evidently destroying one another. Also

$\frac{1+c}{2}\int_{-1}^{+1}\frac{dy}{\Delta(c,y)}=\int_0^1\frac{dx}{\Delta(k,x)}=\mathrm{K}$, and $\log\left(\frac{1-c}{1+c}\right)=\frac{1}{2}\log(1-k^2)$.

Therefore

$$\int_0^1\frac{dx}{\Delta(k,x)}\log(1-k^2x^2)=-\frac{\mathrm{K}}{2}\log(1-k^2)\ldots\ldots\ldots(2).$$

Again, make $x=\left(\frac{1-y^2}{1-k^2y^2}\right)^{\frac{1}{2}}$; then we have $\frac{dx}{\Delta(k,x)}=-\frac{dy}{\Delta(k,y)}$.

Also,

$$\int_0^1\frac{dx}{\Delta(k,x)}\log x=-\tfrac{1}{2}\int_1^0\frac{dy}{\Delta(k,y)}\log(1-y^2)+\tfrac{1}{2}\int_1^0\frac{dy}{\Delta(k,y)}\log(1-k^2y^2)$$

$$=\tfrac{1}{2}\int_0^1\frac{dy}{\Delta(k,y)}\log(1-y^2)-\tfrac{1}{2}\int_0^1\frac{dy}{\Delta(k,y)}\log(1-k^2y^2).$$

Change y into x in the second member, and put for the integral in the first member, and the last integral in the second, their values from (1) and (2); it is easily found that

$$\int_0^1\frac{dx}{\Delta(k,x)}\log(1-x^2)=\frac{\mathrm{K}}{2}\log\left(\frac{1-k^2}{k^2}\right)-\frac{\pi}{2}\mathrm{K}'\ldots\ldots(3).$$

Make now $x=\frac{(1+c)y}{1+cy^2}$; then $\frac{dx}{\Delta(k,x)}=(1+c)\frac{dy}{\Delta(c,y)}$, and

$$\int_0^1\frac{dx}{\Delta(k,x)}\log x=(1+c)\int_0^1\frac{dy}{\Delta(c,y)}\log(1+c)+(1+c)\int_0^1\frac{dy}{\Delta(c,y)}\log y-$$

$$(1+c)\int_0^1\frac{dy}{\Delta(c,y)}\log(1+cy^2).$$

Suppose $\int_0^1\frac{dy}{\Delta(c,y)}=\mathrm{K}_1$, $\int_0^1\frac{dy}{\Delta(c',y)}=\mathrm{K}'_1$, $c^2+c'^2=1$; then $\mathrm{K}=(1+c)\mathrm{K}_1$, $\frac{\mathrm{K}'_1}{1+k'}=\frac{1+c}{2}\mathrm{K}_1'=\mathrm{K}'$. Putting now for the three first of the above integrals their values, and eliminating K, we find

$$\int_0^1\frac{dy}{\Delta(c,y)}\log(1+cy^2)=-\frac{\pi}{8}\mathrm{K}'_1-\frac{\mathrm{K}_1}{2}\log\left(\frac{c^{\frac{1}{2}}}{2(1+c)}\right).$$

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Change y into x, c into k, K_1 into K, and K'_1 into K_1; then

$$\int_0^1 \frac{dx}{\Delta(k,x)} \log(1+kx^2) = -\frac{\pi}{8} K' - \frac{K}{2} \log\left(\frac{k^{\frac{1}{2}}}{2(1+k)}\right) \ldots\ldots\ldots (4).$$

This transformation gives $1 - k^2x^2 = \left(\frac{1-cy^2}{1+cy^2}\right)^2$; and therefore

$$\int_0^1 \frac{dy}{\Delta(c,y)} \log\left(\frac{1-cy^2}{1+cy^2}\right) = \frac{1}{2(1+c)} \int_0^1 \frac{dx}{\Delta(k,x)} \log(1-k^2x^2)$$
$$= \frac{K}{4(1+c)} \log(1-k^2) = \frac{K'}{2} \log\left(\frac{1-c}{1+c}\right).$$

Change y into x, c into k, K_1 into K, and the last becomes

$$\int_0^1 \frac{dx}{\Delta(k,x)} \log\left(\frac{1-kx^2}{1+kx^2}\right) = \frac{K}{2} \log\left(\frac{1-k}{1+k}\right).$$

And the last, by means of (4), gives

$$\int_0^1 \frac{dx}{\Delta(k,x)} \log(1-x^2) = -\frac{\pi}{8} K' - \frac{K}{2} \log\left(\frac{k^{\frac{1}{2}}}{2(1-k)}\right) \ldots\ldots (5).$$

The same transformation also gives

$$c^{\frac{1}{2}}y = \left(\frac{1-\sqrt{1-k^2x^2}}{1+\sqrt{1-k^2x^2}}\right)^{\frac{1}{2}} = \frac{1-\sqrt{1-k^2x^2}}{kx} = \frac{kx}{1+\sqrt{1-k^2x^2}}.$$

Therefore

$$\int_0^1 \frac{dx}{\Delta(k,x)} \log(1-\sqrt{1-k^2x^2}) = \int_0^1 \frac{dx}{\Delta(k,x)} \log(kx) + (1+c) \int_0^1 \frac{dy}{\Delta(c,y)} \log(c^{\frac{1}{2}}y).$$

Reducing this by what goes before, we find

$$\int_0^1 \frac{dx}{\Delta(k,x)} \log(1-\sqrt{1-k^2x^2}) = \frac{K}{2} \log k - \frac{3\pi}{4} K' \ldots\ldots\ldots (6).$$

In exactly the same manner we obtain

$$\int_0^1 \frac{dx}{\Delta(k,x)} \log(1+\sqrt{1-k^2x^2}) = \frac{K}{2} \log k + \frac{\pi}{4} K' \ldots\ldots\ldots (7).$$

But $2(1-\sqrt{1-k^2x^2}) = (\sqrt{1+kx} - \sqrt{1-kx})^2$

$2(1+\sqrt{1-k^2x^2}) = (\sqrt{1+kx} + \sqrt{1-kx})^2$, and $\frac{1}{2}\int_0^1 \frac{dx}{\Delta(k,x)} \log 2 = \frac{K}{2} \log 2$.

Therefore the two last will give

$$\int_0^1 \frac{dx}{\Delta(k,x)} \log(\sqrt{1+kx} - \sqrt{1-kx}) = \frac{K}{4} \log(4k) + \frac{3\pi}{8} K' \ldots\ldots (8).$$

$$\int_0^1 \frac{dx}{\Delta(k,x)} \log(\sqrt{1+kx} + \sqrt{1-kx}) = \frac{K}{4} \log(4k) + \frac{\pi}{8} K' \ldots\ldots (9).$$

Make $y = \frac{(1+k')x\sqrt{1-x^2}}{\sqrt{1-k^2x^2}}$. Then also $\frac{dx}{\Delta(k,x)} = \frac{1+c}{2} \cdot \frac{dy}{\Delta(c,y)}$,

and $\sqrt{1-c^2y^2} = \frac{1-(1-k')x^2}{\sqrt{1-k^2x^2}}$. Therefore we shall obtain

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$$\int_0^1 \frac{dx}{\Delta(k,x)} \log\{1-(1-k')x^2\} - \tfrac{1}{2}\int_1^0 \frac{dx}{\Delta(k,x)} \log(1-k^2x^2) =$$
$$\frac{1+c}{4}\int_0^0 \frac{dy}{\Delta(c,y)} \log(1-c^2y^2)^2 = \frac{1+c}{2}\int_0^1 \frac{dy}{\Delta(c,y)} \log(1-c^2y^2).$$

Or,

$$\int_0^1 \frac{dx}{\Delta(k,x)} \log\{1-(1-k')x^2\} = \frac{K}{4}\log(1-k^2) + \frac{K}{4}\log(1-c^2) \ldots (10).$$

The last term of the second member is found by eliminating K_1.

The three following examples, which do not contain an elliptic function, may perhaps be thought not unworthy of notice.

Multiply $(1-k^2x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}k^2x^2 + \frac{1.3}{2.4}k^4x^4 + \frac{1.3.5}{2.4.6}k^6x^6 + etc.$

by $$\arcsin kx = kx + \tfrac{1}{2}\frac{k^3x^3}{3} + \frac{1.3}{2.4}\frac{k^5x^5}{5} + \frac{1.3.5}{2.4.6}\frac{k^7x^7}{7} + etc.,$$

and the product is $kx + \frac{2}{3}k^3x^3 + \frac{2.4}{3.5}k^5x^5 + \frac{2.4.6}{3.5.7}k^7x^7 + etc.$

This multiplied by $\frac{xdx}{\sqrt{1-x^2}}$, and integrated, since $\int_0^1 \frac{x^{2n}dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\frac{1.3.5..(2n-1)}{2.4.6..(2n)}$, gives

$$\frac{\pi}{4}\left\{k + \frac{k^3}{2} + \frac{k^5}{3} + \frac{k^7}{4} + etc.\right\} = -\frac{\pi}{4k}\log(1-k^2).$$ Therefore

$$\int_0^1 \frac{xdx}{\Delta(k,x)} \arcsin kx = -\frac{\pi}{4k}\log(1-k^2) \ldots\ldots (11).$$

But $\int \frac{xdx}{\Delta(k,x)} = \frac{1}{2k}\log\{1+k^2-2k^2x^2-2k\Delta(k,x)\}$. The first member of (11) therefore integrated by parts, gives $\frac{1}{2k}\log\{1+k^2-2k^2x^2-2k\Delta(k,x)\} \times$ arc sin $kx - \frac{1}{2}\int \frac{dx}{\sqrt{1-k^2x^2}} \log\{1+k^2-2k^2x^2-2k\Delta(k,x)\}$. And this, between the limits $x=0$, $x=1$, is $\frac{1}{2k}$ arc sin $k \log(1-k^2)$. Therefore

$$\int_0^1 \frac{dx}{\sqrt{1-k^2x^2}} \log\{1+k^2-2k^2x^2-2k\Delta(k,x)\} = \frac{1}{k}\left\{\arcsin k + \frac{\pi}{2}\right\}\log(1-k^2)$$
$\ldots\ldots (12).$

And, since $1+k^2-2k^2x^2-2k\Delta(k,x) = \frac{(1-k^2)^2}{1+k^2-2k^2x^2+2k\Delta(k,x)}$, we get

$$\int_0^1 \frac{dx}{\sqrt{1-k^2x^2}} \log\{1+k^2-2k^2x^2+2k\Delta(k,x)\} = \frac{1}{k}\left\{\arcsin k - \frac{\pi}{2}\right\}\log(1-k^2)$$
$\ldots\ldots (13).$

If we make $x = \frac{y}{\sqrt{1+y^2}}$, $k^2 = 1-k^2$; then $y = \frac{x}{\sqrt{1-x^2}}$,

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$\sqrt{1+y^2}=\frac{1}{\sqrt{1-x^2}}$, $\sqrt{1+h^2y^2}=\frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}$, and $\frac{dy}{\Delta'(h,y)}=(1+k)\frac{dx}{\Delta(k,x)}$,

where $\Delta'(h,y)=\sqrt{(1+y^2)(1+h^2y^2)}$.

By means of these the integrals $\int_0^\infty \frac{dy}{\Delta(h,y)}\log y$, $\int_0^\infty \frac{dy}{\Delta(h,y)}\log(1+y^2)$, $\int_0^\infty \frac{dy}{\Delta(h,y)}\log(1+h^2y^2)$, *etc.* are obviously given.

The assumption $y=\frac{(1+k)x}{\Delta(k,x)}$, $h=\frac{1-k}{1+k}$, gives $\frac{dy}{\Delta'(h,y)}=(1+k)\frac{dx}{\Delta(k,x)}$; which perhaps may give integrals of new forms. And since we can express $\frac{dK}{dk}$ in terms of K and the elliptic function of the second species, and $\frac{dK'}{dk}=\frac{dK'}{dk'}\frac{dK'}{dk}$, we may differentiate the integrals already obtained relative to k; and we may employ many other transformations.

Gunthwaite Hall, near Barnsley,
Dec. 7th, 1846.

ON THE DIVISION OF AN ELLIPSOID INTO EQUAL VOLUMES.

[*Mr. Thomas Weddle, Newcastle-upon-Tyne.*]

The object of the following paper is the discussion of the problem,

To divide a given ellipsoid into n equal volumes,

and as I am not aware that this has been attempted before, I trust the subject will be found to possess some degree of interest.

I shall first dispose of two or three particular solutions which it will be advantageous to consider independently of the general method. One of these solutions is as follows:

I. Draw any diameter A′OA of the ellipsoid, and a diametral plane conjugate to it; the section of the ellipsoid by this plane will be an ellipse; by (D) of my paper entitled "Properties of the Ellipse," Mathematician, Vol. II. p. 181, divide this ellipse into n equal sectors P_1OP_2, P_2OP_3.... $P_{n-1}OP_n$, P_nOP_1; through A′A and each of the lines OP_1, OP_2.... OP_n, let a plane be made to pass, then will the ellipsoid be divided into n equal wedges by these n planes.

This does not require a formal demonstration; its truth is evident from the consideration that each plane section of the ellipsoid parallel to that of the diametral plane is divided into n equal sectors by its intersection with the n planes.

Moreover the preceding construction is not confined to an entire ellipsoid; it will apply to any finite portion (cut off by parallel planes) of an ellipsoid, an hyperboloid of one or two sheets, an elliptic paraboloid, a cone or an elliptic cylinder. We may here proceed in the following manner.

I′. Since the volume cut off is finite, the sections made by the parallel planes are necessarily ellipses; divide either of these into n equal sectors P_1OP_2.... P_nOP_1, (as above); find the diameter conjugate to the parallel

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planes (which may be done by joining the centres O', O, of the sections), and through O'O and the lines OP_1, OP_2,....OP_n let planes pass, these will divide the solid into n equal wedges.

It will not fail to be observed that the constructions I and I' will still apply when the problem is to divide the solid into portions having given ratios to each other, provided the ellipse be divided into sectors having these ratios instead of into equal parts. These processes also hold with respect to portions of the solids cut off by planes passing through A'A (or O'O), the figure to be divided into equal parts or into parts having given ratios, being now a sector of an ellipse. (See (T) Mathematician, Vol II. p. 186.)

Of course none of the preceding operations will be *geometrical* unless the division of the ellipse (or elliptic sector, as the case may be) can be performed geometrically. A similar remark is also applicable to some of the succeeding solutions.

Before proceeding to the next method, it will be necessary to premise the following lemmas (α) and (β).

(α). In A'A any diameter of an ellipsoid, take any point P, through which let a plane be drawn parallel to the diametral plane conjugate to A'A: with the centre O as vertex and curve of section as director, let a cone be described; this conical surface will cut off a solid sector which is to the whole ellipsoid as AP is to A'A.

Take any two diameters B'B, C'C (which will be in the diametral plane) forming with A'A a system of conjugate diameters. Denote AO, BO, CO, and PO by a, b, c and k. The volume of the whole ellipsoid is $\frac{4}{3}\pi m abc$, (m being a number which depends on the inclinations of A'A, B'B and C'C to each other), and it is easy to see that the volume of that part of the cone cut off by the plane passing through P is $\frac{1}{3}\pi mbc\left(k-\frac{k^3}{a^2}\right)$; also the volume of the part of the ellipsoid cut off by the same is $\pi m bc \int_k^a \left(1-\frac{x^2}{a^2}\right)dx =$ $\frac{1}{3}\pi m bc.\left\{2a-3k+\frac{k^3}{a^2}\right\}$; hence adding the volumes of the cone and segment to get that of the solid sector cut off by the conical surface, we have,

solid sector $= \frac{2}{3}\pi m bc(a-k)$;

$\therefore$ solid sector : whole ellipsoid : : $\frac{2}{3}\pi m bc(a-k) : \frac{4}{3}\pi mabc$

: : $a-k$: $2a$: : AP : A'A.

It also easily follows from (α), that,

(β). If through two points, P and Q, in a diameter A'A, planes be drawn and conical surfaces be described as in (α), the portion of the ellipsoid included between these two surfaces is to the whole ellipsoid as PQ : A'A.

We are now prepared for our second solution.

II. Take any diameter A'A of the ellipsoid, divide it into n equal parts, and through the $n-1$ points of division let planes be drawn parallel to the diametral plane conjugate to A'A; with the centre as common vertex and the curves in which these planes cut the ellipsoid as directors, let $n-1$ conical surfaces be described; then will these surfaces divide the ellipsoid

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into n equal parts. If A'A, instead of being divided into equal parts, be divided into parts having any ratios to each other, the ellipsoid will be divided into volumes having these ratios.

This construction, the truth of which is evident from (α, β), is very simple and general; it requires, however, the description of cones of the second order, so that it is not strictly geometrical.

The preceding solution will also apply to a portion of an ellipsoid cut off by parallel planes, as follows:

II'. Divide the line joining the centres of the sections into n equal parts or into n parts having the given ratios, and draw planes parallel to those of the sections and describe cones as in II.

A little consideration will make it evident that if a wedge be cut off from an ellipsoid by two planes passing through A'A, we may vary (α) by saying,—the corresponding portion of the solid sector is to the wedge in the same ratio as before, namely, AP to A'A; hence also, instead of (β) we shall have, the portion of the wedge included between the two cones is to the whole wedge as PQ is to A'A. From these considerations it appears that the process II will effect the division of a wedge cut off by planes passing through the diameter employed, into equal parts, (or parts having given ratios), and hence, when n is a composite number $= pq$, another method of dividing an ellipsoid into n equal volumes may be got by combining I and II in the following manner.

III. Divide the ellipsoid into p equal parts by I, and into q equal parts by II, using the same diameter A'A in both operations; then, by the mutual intersections of the p planes, and $q-1$ cones, the ellipsoid will be divided into pq equal volumes.

In the paper alluded to in I, I have virtually shown that the division of an ellipse into equal sectors may be made to depend on that of the circumference of a circle into equal arcs; I shall now endeavour to establish the analogous property in space, namely, that the division of an ellipsoid into equal volumes may be effected when the surface of a sphere can be divided into as many equal surfaces.

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and $\frac{x^2}{d^2} + \frac{y^2}{e^2} + \frac{z^2}{f^2} = 1$, be the equations to two ellipsoids, (either to the same or different axes); (x_1, y_1, z_1) a point on the surface of the former, and (x_2, y_2, z_2) a point on that of the latter; also let

$$\frac{x_1}{x_2} = \frac{a}{d}, \quad \frac{y_1}{y_2} = \frac{b}{e}, \quad \frac{z_1}{z_2} = \frac{c}{f},$$

the points (x_1, y_1, z_1) and (x_2, y_2, z_2), so related, are called *corresponding points*; also if two points move on the surfaces so as to be always corresponding, they will trace out *corresponding curves*.

From the three preceding equations we see that if the coordinates of one of two corresponding points be given, those of the other will be got by simply finding fourth proportionals, and it will be useful to observe that if one of the points be denoted by (al, bm, cn), where l, m, n are subject to the condition $l^2 + m^2 + n^2 = 1$, the other will be denoted by (dl, em, fn).

If one of two corresponding curves, that on the surface of the first ellipsoid for instance, be a plane curve, the other will be plane also, for if the plane $Ldx + Mey + Nfz = P$, contain the point (al, bm, cn), the plane

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$Lax + Mby + Ncz = P$, will contain the point (dl, em, fn), for the condition in each case is $Ladl + Mbem + Ncfn = P$.

Hence if any plane section of an ellipsoid be given, the corresponding curve on any other ellipsoid may be found. Take three points on the given curve and find the corresponding points on the other ellipsoid; the plane passing through these three points will intersect the ellipsoid in the curve required. Moreover, if the plane of one of two corresponding plane curves pass through the centre of the ellipsoid on which it is situated, we shall have $P = 0$, in the preceding equations, and consequently the plane of the second curve will pass through the centre of the other ellipsoid. Hence, as in this case we already know one point in the plane of the second curve, only two corresponding points will be required to construct it; these two points must not however be diametrically opposite.

Finally, we must establish the following theorem.

(γ). If corresponding closed curves* on two ellipsoids be the directors of two curves, having their vertices at the respective centres, the solid sectors cut off by the conical surfaces will be proportional to the ellipsoids themselves.†

Let the portions of the surfaces of the ellipsoids contained within the curves be covered with an immense number of corresponding points. Let P, Q, R be three adjacent points on one of the surfaces, and p, q, r the corresponding points on the other. The plane triangles PQR, pqr will, on account of their extreme smallness, nearly coincide with the surfaces, and the triangular pyramids OPQR, $opqr$, (O and o being the centres of the ellipsoids), may be taken as the elements of the solid sectors whose volumes, V and v, we wish to compare. Denote the points P,Q,R by (al_1, bm_1, cn_1), (al_2, bm_2, cn_2), (al_3, bm_3, cn_3), then the corresponding points p, q, r will be denoted by (dl_1, em_1, fn_1), (dl_2, em_2, fn_2), (dl_3, em_3, fn_3). Hence (Young's Anal. Geom., Vol. II, p. 243),

$$\delta V = \text{pyramid OPQR} = \frac{1}{6} abc\{(l_2m_3 - l_3m_2)n_1 + (l_3m_1 - l_1m_3)n_2 + (l_1m_2 - l_2m_1)n_3\},$$

and

$$\delta v = \text{pyramid } opqr = \frac{1}{6} def\{(l_2m_3 - l_3m_2)n_1 + (l_3m_1 - l_1m_3)n_2 + (l_1m_2 - l_2m_1)n_3\},$$

$$\therefore \frac{\delta V}{abc} = \frac{\delta v}{def}, \text{ and integrating } \frac{V}{abc} = \frac{v}{def},$$

which establishes the proposition, for the ratio of the ellipsoids is the same as $abc : def$.

If one of the ellipsoids be a sphere, and its surface be divided by curves into n surfaces equal in area, and cones having these curves for directors and the centre for common vertex be described, then will the n solid sectors into which the sphere is thus divided be equal to each other. It hence

* It must be borne in mind that these curves may be discontinuous and then the cones will be so too. Also, if any portion of such a curve be contained in a plane passing through the centre of the ellipsoid, a part of the *cone*, having the curve for director and its vertex at the centre, will coincide with a portion of this plane.

† I find that this theorem has been previously given (as a lemma) by Mr. Jellet in a paper "On the Quadrature of Surfaces of the Second Order." See the Camb. and Dublin Math. Journal, new series, No. I, p. 59.

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follows, with the aid of (γ), that if the curves on the ellipsoid corresponding to those on the sphere be found, the ellipsoid will be divided into n equal volumes by the description of cones having these curves for directors and the centre for common vertex. Now, if the curves dividing the spherical surface be plane, that is, the arcs of either great or small circles, we know from what has been previously shown that the corresponding curves on the ellipsoid can be described, and the planes containing the latter curves will evidently either pass through the centre or not, according as great or small circles are employed. Moreover it is clear that in the former case, the dividing *cones* will consist of portions of *planes* only, and hence the construction will be effected geometrically, which however, cannot be done when small circles are used.

Returning now to the problem which it is the object of this paper to discuss, we shall evidently have the following

GENERAL SOLUTION.

(A). Divide the surface of a sphere, in any manner, into n equal surfaces by arcs of great or small circles, find the corresponding curves on the surface of the ellipsoid, and with these curves as directors and the centre of the ellipsoid as common vertex, describe cones; these will, by their mutual intersections, divide the ellipsoid into n equal volumes.

If small circles be excluded, the preceding process may be modified as follows, and it may now be termed the

GENERAL GEOMETRICAL SOLUTION.

(B). Divide the surface of a sphere into n equal surfaces by the arcs of great circles, find the points on the ellipsoid corresponding to the angles of the spherical figures into which the surface of the sphere is divided, and join these points by straight lines in the same order as the corresponding points on the sphere are by the arcs of great circles; through each of these straight lines and the centre of the ellipsoid, let planes be drawn; then will the ellipsoid be divided into n equal volumes.

N.B. Should any of the great circle arcs be a semicircle, the last construction, determining merely two diametrically opposite points in the corresponding curve on the ellipsoid, would be insufficient. In this case we must take a point in the semicircle and treat it as one of the angles of the spherical figures.

It is evident that an ellipsoid may be divided into parts having given ratios to each other, by means of (A) and (B), providing the surface of the sphere be divided into parts having these ratios to each other instead of into equal parts. Also, if a closed curve, either contained entirely in one plane or consisting of portions of plane curves, be traced on the surface of an ellipsoid, the solid cut off by the conical surface having its vertex at the centre and the curve for director, may be divided into equal volumes, or into volumes having given ratios, providing the spherical figure bounded by the corresponding curve on a sphere, can be similarly divided. This includes as a particular case, an ellipsoidal wedge, that is, a portion of an ellipsoid cut off by two diametral planes, the spherical figure to be divided being a lune.

All that now remains to be done in order to complete the solution of our problem, is merely to notice the most interesting methods of dividing a spherical surface into equal parts, and this I shall dispose of as briefly as possible.

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1. The surface of a sphere may be divided into n equal parts as follows: Take any circle, either great or small, of the sphere, and inscribe in it a regular polygon of n sides, through the poles of this circle and each of the angles of the polygon, draw planes intersecting the sphere in n semicircles or meridians, then will the surface of the sphere be divided into n equal lunes.

2. Take any diameter of the sphere, divide it into n equal parts, and through the $n-1$ points of division draw planes perpendicular to the diameter; these planes will divide the surface into n equal parts. (See Hymers's Int. Cal., third edition, p. 182.)

3. If n be a composite number $= pq$, we may combine the preceding constructions. Divide the surface of the sphere into p equal parts by (1), and into q equal parts by (2), using in the latter operation the diameter passing through the poles of the circle employed in the first. The mutual intersections of the great and small circles will divide the surface of the sphere into pq equal parts.

4. Another obvious and interesting, though very limited, method is by inscribing a regular body in the sphere, and joining the angular points by arcs of great circles in the same order as the edges of the body. In this way we may divide the surface into 4, 6, 8, 12, or 20 equal and regular spherical figures; and if great circle arcs be drawn from the angles of each figure to the near pole of its circumscribing circle, the surface of the sphere will be divided into 12, 24, 24, 60, or 60 equal isosceles triangles respectively, and the number of divisions may be doubled by bisecting the vertical angle of each triangle.

5. Divide the sphere into two hemispheres, and in each describe $\frac{1}{2}n-1$, (n even), or $\frac{1}{2}(n-1)$, (n odd), spherical polygons, whose areas shall be equal to θ, 2θ, 3θ, *etc.* respectively, (θ being the n^{th} part of the surface of the sphere), and let the polygons of each hemisphere be described in such a manner that the m^{th} polygon shall contain all the preceding $m-1$ polygons; then, accounting the great circle dividing the hemispheres a polygon when n is even, but not when n is odd, the areas of the surfaces included between the polygons will be equal, and hence the surface of the sphere will be divided into n equal surfaces.

The neatest mode of proceeding will be to describe regular spherical polygons of the same number of sides in such a way that the poles of the small circles circumscribing them may coincide with those of the great circle dividing the hemispheres. I shall now show how these polygons may be constructed.

In the first place we may observe, that if the angles of a spherical triangle be given, it may be described, for the sides and angles of the polar triangle are respectively equal to the supplements of the angles and sides of the primitive triangle; hence the polar triangle may be constructed and the supplements of the measures of its angles will be the sides of the primitive triangle.

Let it now be required to construct a regular spherical polygon of m sides whose area shall be ϕ. Suppose AB to be a side of the polygon, and O the near pole of its circumscribing circle. The area of the isosceles triangle AOB evidently $= \frac{\phi}{m}$ and $\sphericalangle O = \frac{2\pi}{m}$, hence $\sphericalangle A + \sphericalangle B + \sphericalangle O - \pi = \frac{\phi}{m}$, or $A = B = \frac{\pi}{2} + \frac{\phi}{2m} - \frac{\pi}{m}$; the angles of the triangle AOB being now

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known, the polar triangle aob must be constructed, and we shall then know $AB = \pi - < O$, and $AO = BO = \pi - \sphericalangle a$; a side of the regular spherical polygon, and the radius of its circumscribing circle being thus found, we are obviously enabled to describe the polygon.

Hence we have only to take $\phi = \frac{4\pi}{n}, \frac{8\pi}{n} \ldots\ldots \frac{4p\pi}{n}$ in succession, (here $p = \frac{1}{2}n - 1$, when n is even, and $= \frac{1}{2}(n-1)$, when n is odd), and describe the resulting polygons in each hemisphere, making the poles of their circumscribing circles coincide with those of the great circle dividing the hemispheres; then, accounting as above, the great circle a polygon, or not, according as n is even or odd, the surface will be divided into n equal surfaces.

6. If n be a composite number $= pq$, we may combine (1) and (5).

Inscribe a regular (plane) polygon of p sides in any great circle, and through its poles L and M, and the angles of the polygons pass planes as in (1), which will intersect the sphere in p semicircles or meridians. Again, find as in (5), the radii of the small circles circumscribing those regular spherical polygons of p sides that are capable of dividing the spherical surface into q equal parts. With L and M as poles, and these radii, describe small circles, and join (in order) the p points in which each circle intersects the meridians, by great circle arcs; these arcs and the meridians will divide the surface of the sphere into pq equal surfaces.

It must be borne in mind that if n be even, the great circle first taken will supply the place of a small circle.

Several other methods of dividing the surface of a sphere into equal surfaces might be given, but these are the most interesting ways that I have been able to think of. The most of them are new so far as is known to me, though it seems very probable that several of them will prove to be old. It will not fail to be observed that all the methods, except (4), may be adapted to divide a spherical lune into equal surfaces, and that (1), (2), and (5) may be modified so as to divide a spherical surface or a spherical lune into surfaces having given ratios.

The methods of dividing an ellipsoid into equal volumes that may be got by combining (1), (2), and (3) with (A) or (B), have already been given under better forms, I, II, III. The solutions afforded by (B), with the aid of (5) and (6), might also be simplified, though not very materially, and hence, as this paper is already much longer than I anticipated, I shall not dwell upon these modifications.

I shall, in conclusion, advert to a mode of dividing the ellipsoid which is altogether different from any of the preceding. It is true, that this solution is never geometrical (except $n = 2$) as it depends on the trisection of an angle, while most of those previously given are, in certain cases, capable of being performed geometrically; nevertheless, this method is so simple and elegant, that I am persuaded I ought not to omit it.

Let A′A be any diameter of the ellipsoid, in which take the centre O and any point S, and through S draw a plane conjugate to A′A cutting off a segment from the ellipsoid. If $OA = a$, and $OS = x$, it may readily be shown that $x^3 - 3a^2x + 2a^3 : 4a^3 ::$ segment : whole ellipsoid $:: h : 1$ (suppose), hence $x^3 - 3a^2x = 2a^3(2h - 1)$; let $x = 2a\cos\theta$, then this equation becomes $4\cos^3\theta - 3\cos\theta = 2h - 1$, or $\cos 3\theta = 2h - 1$, $\therefore 3\theta = \cos^{-1}(2h - 1)$.

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Now if a be the smallest positive value of $\cos^{-1}(2h-1)$, (so that $a > 0 < \pi$), the only values of 3θ that need be considered are a, $2\pi+a$, and $2\pi-a$; but since x or $2a\cos\theta$ necessarily $\gtrless -a \lessgtr a$, it is easy to see that $\theta > \frac{1}{3}\pi < \frac{2}{3}\pi$ or $3\theta \gtrless \pi \lessgtr 2\pi$, and hence $2\pi-a$ is the only admissible value of 3θ. If then by $\cos^{-1}(2h-1)$ we understand that particular value between 0 and π, we shall have $\theta = \frac{1}{3}\{2\pi-\cos^{-1}(2h-1)\}$, and thus OS or $x=2a\cos\theta$ is known when h is given.

Hence the following method of dividing an ellipsoid into n equal parts.

IV. Through any diameter A'OA pass a plane, in which with the centre O and radius = A'A, describe a circle KMLN meeting A'A produced in K and D. Divide KL into n equal parts in the points $P_1, P_2 \ldots\ldots P_{n-1}$, and through these points draw perpendiculars to KL meeting the semicircle KML in $Q_1, Q_2 \ldots\ldots Q_{n-1}$. Trisect the arcs LNQ_1, $LNQ_2 \ldots\ldots LNQ_{n-1}$ in $R_1, R_2 \ldots\ldots R_{n-1}$, so that LR_1, $LR_2 \ldots\ldots LR_{n-1}$ may be one third of these arcs; from $R_1, R_2 \ldots\ldots R_{n-1}$ draw perpendiculars to A'A meeting it in $S_1, S_2 \ldots\ldots S_{n-1}$, and, finally, through $S_1, S_2 \ldots\ldots S_{n-1}$ pass planes conjugate to A'A; these divide the ellipsoid into n equal volumes.

For $KL = 2A'A = 4a$, $LP_m = \frac{4(n-m)a}{n}$, and $OP_m = LP_m - LO = 2a\left(1-\frac{2m}{n}\right)$, hence arc $LQ_m = 2a.\cos^{-1}\frac{-OP_m}{2a} = 2a.\cos^{-1}\left(\frac{2m}{n}-1\right)$, arc $LNQ_m = 2a.\left\{2\pi - \cos^{-1}\left(\frac{2m}{n}-1\right)\right\}$, and arc $LR_m = 2a \times \frac{1}{3}\left\{2\pi-\cos^{-1}\left(\frac{2m}{n}-1\right)\right\}$, wherefore $OS_m = 2a.\cos\frac{\text{arc } LR_m}{2a} = 2a.\cos\frac{1}{3}\left\{2\pi - \cos^{-1}\left(\frac{2m}{n}-1\right)\right\}$. It thus appears from what has been said above, that the segment cut off by the plane passing through S_m is to the whole ellipsoid as $m : n$; hence the successive segments are proportional to $1, 2, \ldots\ldots n-1$, and their successive differences or the parts into which the ellipsoid is divided, are equal to each other.

A similar process will evidently effect the division of an ellipsoid into parts having given ratios to each other. Also a wedge cut off by two diametral planes will be divided into equal volumes or into volumes having given ratios, in the very same manner as the whole ellipsoid.

When n is a composite number $=pq$, we may combine I and IV as follows:

V. Divide the ellipsoid into p equal parts by I, and into q equal parts by IV, using the same diameter in both cases. The mutual intersections of the various planes will divide the ellipsoid into pq equal volumes.

It might also be shown that the division of a portion of an hyperboloid of one or two sheets, cut off by parallel planes, may be effected by the solution of a number of cubic equations, the geometrical construction of which would generally require us to be able not only to trisect an angle, as in IV, but to insert two mean proportionals between given lines. Also since the portion of an elliptic paraboloid cut off by any plane conjugate to a given diameter is proportional to the square of the segment of the diameter intercepted between the surface and the plane, the division of an elliptic paraboloid may be effected *geometrically*. The great length, however, to which the discussion of the problem I proposed to myself has extended, precludes me from doing more than hinting at these processes.

November 4th, 1846.

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MODERN GEOMETRY.

[*T. S. Davies, F.R.S.*]

(Continued from page 203.)

RULE-SURFACES OF THE SECOND ORDER.

XXXVIII.

When in any one of three given directrices a point is given, it is required to construct the generatrix through that point.

Let L, L_1, L_2 be the three given directrices, and p a point in one of them, L; and through p and L_1 draw the plane p_1q_1, and through p and L_2 the plane p_2q_2.

Now since the planes p_1q_1, p_2q_2 pass through the point p, they intersect in a straight line passing through that point. That line, therefore, rests upon the directrix L.

Again, since the common section of the planes p_1q_1, p_2q_2, is in *each* of the planes, it will cut or be parallel to all lines whatever, L_1, L_2 situated in those planes; and hence will either cut or be parallel to L_1, L_2. It cannot be parallel to both, since, by hypothesis, those lines are themselves not parallel. It must therefore cut one of them; and similarly, it will cut or be parallel to the other.

The condition, therefore, of p_1pp_2 being a generatrix is fulfilled.

Conversely, we may always consider a generatrix as the intersection of two planes drawn both through a given point in one directrix, and passing through the other two, one plane through one directrix and the other through the other.

XXXIX.

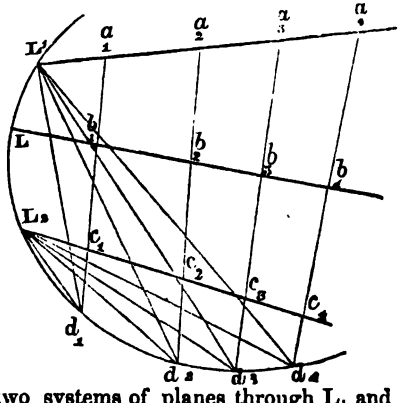
The preceding article shews that the following genesis of the rule-surface known as the hyperboloid of one sheet, does not differ, except in the form of its enunciation, from that which was originally given in (XXVI).

If about two of three straight lines, no two of which are in the same plane, nor all three parallel to the same plane, planes be made to revolve so as to always intersect in the third line: the intersections of these planes will generate the hyperboloid of one sheet.

XL.

Any plane section of the hyperboloid of one sheet, not coinciding with a generatrix or directrix, will be a conic section.

Let L, L_1, L_2 be the three directrices, and in L take any four points b_1, b_2, b_3, b_4; and let the section-plane cut the two other directrices in L_1, L_2, and the generatrices in d_1, d_2, d_3, d_4.



Then, the four planes $L_1a_1b_1$, $L_1a_2b_2$, $L_1a_3b_3$, $L_1a_4b_4$, being cut by the line L in b_1, b_2, b_3, b_4, the anharmonic ratio of the sines of the angles which they form is the same as that of the four points b_1, b_2, b_3, b_4. And in like manner, the anharmonic ratio of the sines of the angles contained by the four planes $L_2c_1b_1$, $L_2c_2b_2$, $L_2c_3b_3$, $L_2c_4b_4$ is the same as that of the four points b_1, b_2, b_3, b_4. Wherefore the two systems of planes through L_1 and L_2 have the same anharmonic ratio subsisting amongst the sines of their contained dihedral angles.

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Now the first system of planes through L_1, is cut by the section-plane in the four lines L_1d_1, L_1d_2, L_1d_3, L_1d_4; and hence (II. 3.) the anharmonic ratio of the sines of the angles which they form is the same with that of the four planes themselves. Similarly the four lines L_2d_1, L_2d_2, L_2d_3, L_2d_4 contain angles whose sines have the same anharmonic ratio as the sines of the dihedral angles themselves.

The system of angles at L_1 has therefore the same anharmonic ratios subsisting between the sines, as the system of angles at L_2. The six points L_1, L_2, d_1, d_2, d_3, d_4 are therefore situated in the same conic section (XVII).

Now the four generatrices are taken arbitrarily as to special position; and hence if we define the conic section by any five of the points (L_1, L_2 being always two of them) as L_1, L_2, d_1, d_2, d_3, the sixth point d_4 will always fall in the perimeter of that conic section.

This proves that the intersection of every generatrix with the section-plane lies in the same conic section, and proves the enunciated proposition.

Again, if we consider the case just discussed to refer to the first mode of genesis, we can in like manner shew that viewing the surface as described by the second mode, the conic section will be the same.

For, by the preceding part of our reasoning, the selected points b_1, b_2, b_3, b_4 were *any whatever* in the line L. One of them, consequently might have been considered as the point L in which the line meets the section-plane, which would also have been in the conic section. Taking, therefore, a_1d_1, a_2d_2 as two of the directrices of the second system of genesis, L, L_1, L_2 would have been three of the corresponding generatrices. Any fourth line, then, moving upon a_1d_1, a_2d_2 and a_3d_3 will describe upon the section-plane the conic section $LL_1L_2d_1d_2$ which is identical with that already spoken of through $L_1L_2d_1d_2d_3$.

It follows, hence, that all the lines of both systems intersect the section-plane in the same conic section: or in other words—

Every plane section of the hyperboloid of one sheet which does not coincide with one of the generating lines of either system, is a conic section.

XLI.

When the section-plane coincides with one of the directrices it must also coincide with some one of the generatrices, and can only coincide with one.

For as every one of the generatrices meets that directrix, there is no one of them which is not in a plane with that directrix. The plane, therefore, being conceived to revolve about the directrix, it will come into coincidence with every generatrix successively. Whence through any directrix, innumerable planes may be drawn that shall cut the surface in a second straight line.

Nor can any plane be drawn through a directrix that shall not cut the surface in a second straight line.

For, one of the generatrices of the surface is situated upon *every plane* drawn through that directrix and a point in a second directrix; and these include all planes drawn through the first directrix upon which one of the generatrices is not placed.

The conic section, then, in this case, is reduced to two straight lines. These lines will meet if the plane do not pass through the centre; and they will be parallel if it do.

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Moreover, as no two generatrices can be parallel, no more than one of them can be in the same given plane through a directrix.

XLII.

Another property is this:—

If lines be drawn parallel to all the generating lines of either system, they will constitute the ordinary cone either right or oblique; and to whichever system these parallels be drawn, the cone will be the same, if drawn through the same point in space as its vertex.

We must premise the following lemma:—

If four planes pass through the same line, and parallel to these four other planes be drawn through a fixed point, all these planes will intersect in a line drawn through that point parallel to the first line.

Let the planes A, B, C, D intersect in a line L, and through a point p let a, b, c, d be drawn respectively parallel to A, B, C, D. Then since A is parallel to a and B to b, they intersect in parallel lines L and l; and as a, b pass through p, the line l also passes through p. In like manner a and c, a and d, pass through l. Whence, &c.

The demonstration of the proposition follows readily from this lemma, thus:—

Through the given point let planes be drawn parallel to $L_1a_1b_1$, $L_1a_2b_2$, $L_1a_3b_3$, $L_1a_4b_4$: these will meet in a line l_1 parallel to L_1. In like manner planes through the given point parallel to $L_2c_1b_1$, $L_2c_2b_2$, $L_2c_3b_3$, $L_2c_4b_4$ will meet in a line l_2 parallel to L_2.

Now the planes through l_1 being parallel to those through L_1, their respective inclinations will be the same in both cases, and hence the anharmonic ratios of their sines will be identical in the composition of their several terms. Similarly those of the terms of the planes through l_2 and L_2 will be identical.

Moreover, the pairs of planes through l_1 and l_2 will have their intersections parallel to the corresponding intersections of those through L_1 and L_2; and hence whatever other character the cone generated by these lines may have, it is generated, as stated in the enunciation, by lines through the given point parallel to the generatrices of the hyperboloid.

Again, the eight planes through L_1 and L_2 being parallel to those through l_1 and l_2, each to each respectively, their traces on the section-plane will be parallel each to each respectively; and the figures traced upon that plane being composed of triangles, those figures are both similar and similarly situated on the section-plane. Whence the curves traced out by the sixth point in each system are also similar and similarly situated: and since the first $L_1LL_2d_1d_2d_3d_4$ is a conic section the other is also a conic section; and the cone itself is the ordinary one, either right or oblique.

When the point through which the lines are drawn parallel to the generatrix is the centre of the hyperboloid, the cone becomes the *asymptotic cone*—it having the same kind of relation to the hyperboloid that the asymptotes have to the hyperbola. If space allow, this subject will be again noticed.* It is sufficient to notice here that the sections of the hyperboloid and cone made by the same plane are concentric, similar, and

* The result here obtained may be deduced in a different manner, (as indeed may, by a corresponding method, the conclusion of XL.) which has much elegance.

Take into consideration two consecutive generatrices of one system: then there are two of the other system parallel to them, and equidistant from the centre, each to each. If the

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similarly situated; and that the systems of conjugate diameters in the two surfaces are identical in position, each to each.

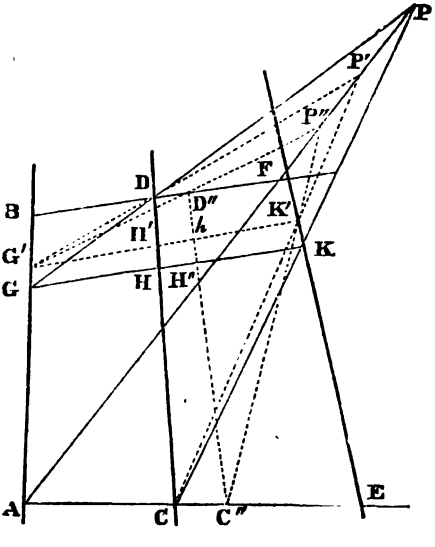
XLIII.

The hyperbolic paraboloid. It now becomes necessary to examine what takes place in the case, which in the preceding inquiry has been formally concluded from our consideration: viz., when *the three given lines are parallel to the same plane.*

The parallelopiped under these circumstances ceases to exist: for instead of the two planes through each line, parallel respectively to the other two lines, we have now only a single plane, which is parallel to both those others. Whence the system is reduced to three parallel planes.

Bearing in mind that no condition has in the general investigation of (XXIX) been imposed upon the quadrilateral, AEFB, except that its opposite sides shall *not be* in one plane, it is obvious that the property existing amongst the segments of its sides will not be affected by the present limitation of the conditions—that the directrices AB, CD, EF are parallel to the same plane. That is, we shall still have

AC.DF : CE.BD
:: AG.KF : GB.EK.



But since AB, CD, EF are parallel to the same plane, they are situated in planes which are also parallel to that plane, and to each other. Whence the generatrices AE, BF, *etc.* always resting on these lines are cut in A,C,E, and B,D,F, by parallel planes; and they are hence divided in the same ratio, and

$$AC : CE :: BD : DF,$$

or $$AC.DF = CE.BD.$$

Wherefore also, we have

$$AG.KF = GB.EK,$$

or $$AG : GB :: EK : KF.$$

It, therefore, follows that the generatrices AE, BF, *etc.* are *themselves parallel to a determinate plane*, the determination being made from *any two* of the generatrices.

planes which contain the two consecutive parallels be drawn they will intersect in a line through the centre, which becomes *ultimately* parallel to both pairs of lines.

Now, since these parallels are generatrices of the first and second modes, they meet the section-plane in the conic section determined in art. (XL.), and hence the planes themselves cut the section-plane in chords of the conic section; and their line of intersection (which is the corresponding edge of the cone) *ultimately* bisects them both. These chords, therefore, bisect each other in the point upon the edge of the cone formed by the intersection of the consecutive planes. Whence *(Leybourn's Repository, N.S. vol.* vi., *p.* 45.) these chords are the tangents to a conic section, concentric, similar, and similarly situated, with that of the hyperboloid.

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When the plane to which the generatrices are parallel is given, two linear directrices only are necessary for fixing their successive positions. For, calling this plane the *directing plane*, and drawing successive planes parallel to it, these planes will cut the directrices each in two points, which determine the successive generating lines.

The surface thus generated is called the *hyperbolic paraboloid* by Hachette, from a property which will hereafter be deduced. It had, however, in the arts received the name of *plan gauche*, or twisted plane; and this is still often employed by the French writers, though not so frequently as Hachette's term.

These two views which may be taken of the genesis of the surface, have led to two different definitions corresponding to them respectively:—

(1) The *hyperbolic paraboloid* is generated by a straight line which moves upon three given directrices, no two of which are in the same plane, but all of which are parallel to one plane:—

(2) The *twisted plane* is generated by a line which moves upon two given straight lines not situated in the same plane, and which is always parallel to a given plane.

It is scarcely necessary to remark that the first of these definitions is the more systematic one, taken in reference to the general investigation of the character of the rule-surfaces whose directrices and generatrices are rectilinear. The second form deduced from it, as is here done, should be considered as a *property*—not as a definition. Yet it is almost, if not altogether, invariably *assumed* by the continental writers without the slightest hint of such a proof as is offered above, that the two definitions are those of identical surfaces:—an assumption which appears to me to be far too remote from obviousness to justify its adoption without a distinct investigation.

It may also, indeed, be remarked that a second definition of the general surface of the hyperboloid of one sheet is almost as frequently employed as that which has been described in this chapter; and that most writers pass from one to the other at pleasure, without in any case taking the trouble (and that is not a slight trouble) to establish the identity of the two geneses, even by a *short* of geometrical reasoning.

XLIV.

Under whichever aspect we view the hyperbolic paraboloid it will follow that it admits of two distinct modes of description, like the hyperboloid of one sheet.

For, viewed as a particular case of the latter surface, the reasoning already employed applies without restriction to the hyperbolic paraboloid. And, again, viewed as the twisted plane, two given directrices and a directing plane being given, any two generatrices will fix a new directing plane for a second genesis, and these generatrices become the directrices of the second mode of description.

It follows, therefore, that through any point in the hyperbolic paraboloid two lines may always be drawn which shall lie wholly upon the surface; that never more than two can be so drawn; that every line of one genesis meets every line of the other, dividing the surface into infinitesimal twisted quadrilaterals; that no line of one genesis can meet another line of the same genesis; *etc.*

The difference, indeed, between the hyperboloid of one sheet and the hyperbolic paraboloid consists wholly in those properties which are dependent on the parallelopiped described upon the three directrices in XXXII. This will be more fully seen hereafter.

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(To be continued.)

ON THE ELLIPSE OF MINIMUM AREA WHICH CAN BE DRAWN TO PASS THROUGH FOUR GIVEN POINTS.

[*Professor Hearn, R. M. College, Sandhurst.*]

Euler, in the *Acta Petrop.* vol. ix., p. 132, has discussed this problem which he characterizes as very difficult. I was surprised on consulting the memoir to find, that on arriving at the final cubic he dismisses the subject without discussing the roots, merely observing that being a cubic it must always have at least one real root. He then discusses the case of the minimum ellipse passing through three given points in a separate memoir, though this latter might have been made to follow from the former in a simple and not inelegant manner. His methods are long and complicated; but as the mere analytical improvement of a process is a merit of very secondary value, I have thought proper to do much more, as an attentive perusal of this paper will abundantly show.

Let ABCD be the quadrilateral of which the angular points are the given points and which must not have a re-entering angle otherwise *no* ellipse could be described about it. Produce two of the opposite sides which are not parallel to meet in E, (the case of the parallelogram will be afterwards considered). Make EAC the axis of x and EBD that of y.

Let $\quad EA = \frac{1}{m}$, $EB = \frac{1}{n}$, $EC = \frac{1}{m'}$, $ED = \frac{1}{n'}$.

The equation of a circumscribing conic section is

$$(mx-1)(m'x-1)+z.xy+(ny-1)(n'y-1)-1=0,$$

in which z the coefficient of xy alone remains arbitrary, and is to be determined by the condition that the area is a minimum. The expanded form of the equation is

$$mm'x^2+z.xy+nn'y^2-(m+m')x-(n+n')y+1=0.....(1),$$

and we have without loss of generality m, m', n, n' all ***positive***, and if moreover $z^2 < 4mm'nn'$ the curve is an ellipse.

If α and β be the coordinates of the centre they satisfy the equations

$$2mm'\alpha+z\beta=m+m'.....(2),$$
$$2nn'\beta+z\alpha=n+n'.....(3).$$

Transform by putting $x+\alpha$ for x, and $y+\beta$ for y, and the centre will be the new origin; the transformed equation being

$$mm'x^2+z.xy+nn'y^2=z.....(4),$$

where $\quad -z=mm'\alpha^2+z.\alpha\beta+nn'\beta^2-(m+m')\alpha-(n+n')\beta+1.$

But by (2) and (3)

$$mm'\alpha^2+z.\alpha\beta+nn'\beta^2=\tfrac{1}{2}(m+m')\alpha+\tfrac{1}{2}(n+n')\beta.$$

Hence $\quad z=\frac{1}{2}(m+m')\alpha+\frac{1}{2}(n+n')\beta-1;$

but also from (2) and (3)

$$\alpha=\frac{2nn'(m+m')-z(n+n')}{b^2-z^2},\quad \beta=\frac{2mm'(n+n')-z(m+m')}{b^2-z^2},$$

where $b^2=4mm'nn'$.

Inserting those in the value of z and reducing

$$z=\frac{(z-a)(z-c)}{b^2-z^2},$$

where $a=mn+m'n'$, $\quad c=mn'+m'n$.

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Now let e be the cosine of the angle E, then r being the distance of any point from the origin

$$r^2 = x^2 + 2exy + y^2,$$

and in order to determine the semi-axes r_1, r_2 we have to find r a maximum or minimum, x and y being subject to the condition (4).

Proceeding in the usual manner the equation is found to be

$$4(z - mm'r^2)(z - nn'r^2) = (r^2 z - 2ze)^2,$$

and the product of the roots of this equation

$$r_1^2 r_2^2 = \frac{4(1-e^2)z^2}{b^2 - z^2}.$$

Hence $$(\text{area})^2 = \pi^2 r_1^2 r_2^2 = 4\pi^2 \sin^2 \mathrm{E} \frac{(z-a)^2(z-c)^2}{(b^2-z^2)^3},$$

by restoring the value of z and putting $\sin^2 \mathrm{E}$ for $1-e^2$.

We have therefore to find $\frac{(z-a)^2(z-c)^2}{(b^2-z^2)^3}$ a minimum, z being $< b$.

Making the differential coefficients with respect to $z = 0$, we have

$$z^3 - 2(a+c)z^2 + (3ac + 2b^2)z - b^2(a+c) = 0 \ldots\ldots \ldots\ldots (5),$$

which will be found to agree with Euler's equation, though it is under a much neater form.

Referring to the values of a, b and c, we have a and c positive, and b may be taken $= + 2\sqrt{mn\, m'n'}$, and therefore positive also,

$$a^2 - b^2 = (mn - m'n')^2, \quad \text{and } c^2 - b^2 = (m'n - mn')^2,$$

which are both positive, and therefore $a > b$; $c > b$.

To separate the roots of (5),

1st, $z = 0$ gives $-b^2(a+c)$, negative,
2nd, $z = b$ gives $3b(a-b)(c-b)$, positive,
3rd, $z = (a+c)$ gives $b^2 + ac - (a^2 + c^2)$.

The last result may be written under either of the forms $b^2 - c^2 + a(c-a)$ or $b^2 - a^2 + c(a-c)$, so that this result is always negative whether $a > c$ or $a < c$.

The roots are therefore all real and lie between the limits $0, b, a+c, +\infty$.

Hence there is one root, and only one, (that between 0 and b) congruent to the proposed problem.

That this value between 0 and b makes the proposed function a *minimum* and not a maximum appears evident from the following. There being but one root of $\frac{du}{dz} = 0$ between 0 and b, there can be but one maximum or minimum value of u between such limits. Now the value $z = 0$ makes $u = \frac{a^2c^2}{b^6}$ a finite positive quantity, and $z = b$ makes $u = +\infty$; consequently the intermediate value corresponding to $\frac{du}{dz}$ must be a *minimum*. There is therefore but one minimum ellipse in every case.

When the quadrilateral is a parallelogram we must refer the curve to the diagonals. We will now have $m' = -m$, $n' = -n$, and our reasoning

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about the roots of (5) does not apply. But we do not require it, for since $a = 2mn$, $c = -2mn$, $b^2 = 4m^2n^2$. Equation (5) reduces to

$$z^3 - b^2z = 0;$$

$\therefore z = 0$, or $z = \pm b$, the first only of which is admissible, and it at once shews that the diagonals of the parallelogram are conjugate diameters.

To deduce from the foregoing the problem of finding the minimum ellipse described through three given points.

If we conceive the points A and B to move up to and coincide with E, then both m and n become infinite, or $\frac{1}{m}$ and $\frac{1}{n}$ each $= 0$. Suppose that in their evanescent state they preserve the ratio $\mu : 1$ or $m = \mu n$.

Then since m' and n' are evanescent in comparison with m and n, we have $a = \mu n^2$, $c = n(\mu n' + m')$, and $b^2 = 4\mu m'n'n^2$.

The equation of the curve also is

$$\mu nm'x^2 + z.xy + nn'y^2 - (\mu n + m')x - (n + n')y + 1 = 0.$$

Now divide by nn' making $\frac{z}{nn'} = \Psi$, and $m' = \mu'n'$, and the equation is

$$\mu\mu'x^2 + \Psi.xy + y^2 - \frac{\mu}{n'}x - \frac{1}{n'}y = 0,$$

and the function to be made a maximum or minimum reduces to

$$\frac{\mu^2}{n'^4} \frac{\{\mu + \mu' - \Psi\}^2}{(4\mu\mu' - \Psi^2)^3},$$

and the equation (5) for z becomes

$$2\Psi^2 - 3(\mu + \mu')\Psi + 4\mu\mu' = 0,$$

which might also have been found by direct differentiation of the modified form of the function.

This quadratic will determine an ellipse such, that being described about the given triangle EDC, and touching a line through E drawn in a given direction, its area shall be a minimum.

For we may immediately determine the roots in the quadratic which are real, and only one of them $< 2\sqrt{\mu\mu'}$, and this is the congruent root.

Now when of all ellipses capable of being described about a given triangle we require that of minimum area, we must also make μ variable. Differentiating with respect to μ, we have therefore

$$\frac{2}{\mu} + \frac{2}{\mu + \mu' - \Psi} - \frac{12\mu'}{4\mu\mu' - \Psi^2} = 0,$$

which combined with the preceding will give

$$\mu = \mu' \text{ and } \Psi = \mu' \text{ or } 2\mu'.$$

The value $\Psi = \mu'$ only is congruent, and μ being equal to μ', shews that the minimum ellipse is such that if a tangent is drawn at E it will be parallel to CD.

Referring also to the general values of the coordinates of the centre, we find they reduce to

$$\alpha = \frac{1}{3m'} \text{ and } \beta = \frac{1}{3n'},$$

so that the centre of the ellipse coincides with the centre of gravity of the triangle.

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I will now give an *à priori* demonstration of the last proposition.

To find the least ellipse which can circumscribe a given triangle.

Assume the equation of the curve under the form

$$pmx(mx-1)+mnxy+qny(ny-1)=0,$$

in which p and q are arbitrary constants, m and n as before.

The expanded equation is

$$pm^2x^2+mnxy+qn^2y^2-pmx-qny=0.$$

If α and β be the coordinates of the centre,

$$2pm^2\alpha+mn\beta=pm, \qquad 2qn^2\beta+mn\alpha=qn,$$

$$\alpha=\frac{2pqmn^2-qn^2m}{4pqm^2n^2-m^2n^2}, \quad \beta=\frac{2pqnm^2-pm^2n}{4pqm^2n^2-m^2n^2}.$$

The equation transformed to the centre is

$$pm^2x^2+mnxy+qn^2y^2=A,$$

$$-A=pm^2\alpha^2+mn\alpha\beta+qn^2\beta^2-pm\alpha-qn\beta;$$

$$\therefore \quad A=\tfrac{1}{2}(pm\alpha+qn\beta).$$

Inserting the values of α and β,

$$A=\frac{pq(p+q+1)}{mn(4pq-1)}.$$

The equation for finding

$$r^2=x^2+2exy+y^2$$

a maximum or minimum is found to be

$$4\left(A-p\frac{m}{n}r^2\right)\left(A-q\frac{n}{m}r^2\right)=\left(r^2-2Ae\right)^2,$$

product of roots $\qquad r_1^2r_2^2=\dfrac{4A^2(1-e^2)}{4pq-1}.$

Hence $\qquad (\text{area})^2=4\pi^2\sin^2E.\ \dfrac{p^2q^2(p+q-1)^2}{(4pq-1)^3}.\ \dfrac{1}{m^2n^2};$

and $\qquad u=\dfrac{p^2q^2(p+q-1)^2}{(4pq-1)^3}$, a maximum or minimum;

$4pq$ must be $\triangleright 1$ in order that u may be positive.

Hence $\dfrac{du}{dp}=0$, and $\dfrac{du}{dq}=0$, give

$$\frac{1}{p}+\frac{1}{p+q-1}=\frac{6q}{4pq-1},$$

$$\frac{1}{q}+\frac{1}{p+q-1}=\frac{6p}{4pq-1}.$$

Subtracting

$$\frac{q-p}{pq}=\frac{6(q-p)}{4pq-1}.$$

Hence $q-p=0$, or $4pq-1=6pq$, or $pq=-\frac{1}{2}$.

But $4pq-1\triangleright 0$, $\therefore pq$ cannot $=-\frac{1}{2}$;

consequently $p=q$ and $\therefore \dfrac{1}{p}+\dfrac{1}{2p-1}=\dfrac{6p}{4p^2-1}$, which gives $p=1$.

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The equation to the required ellipse referred to two of the sides of the triangle is therefore

$$(a)\quad mx(mx-1)+mnxy+ny(ny-1)=0.$$

Also $(\text{area})^2 = \dfrac{4\pi^2 \sin^2 E}{m^2n^2}\cdot\dfrac{1}{3^3}$, and area of triangle $= \dfrac{\sin E}{2mn}$;

$$\therefore \frac{\text{area minimum ellipse}}{\text{area}.\Delta} = \frac{4\pi}{3\sqrt{3}}.$$

N. B. It is to be noted that in this solution the equation to the ellipse (a) is fully brought out, and no eliminations of any consequence have been omitted.

January 11*th*, 1847.

ON THE PECULIAR EQUATION $x^3 - bx = 1$.

[*James Lockhart, Esq.*]

(Continued from page 254.)

A few small values of b are, in the present state of this problem, not suited to the resolution. The effectiveness of $b=6$ may be shewn in the following manner, and as this number answers the purpose, all superior integers and mixed fractions will be equally effective.

Let
$$x^3 - 6x = 1,$$
$$x'^3 + 6x' = 1,$$

and as the equations in p were introduced to render the subject more intelligible, they will now be dispensed with.

Now p and $p' = \dfrac{3s}{r} = \dfrac{2b^2}{3} - \dfrac{2}{b} - \dfrac{3}{b^4 - 6b}$;

$$\therefore p = 23{\cdot}664,\quad p' = 24{\cdot}331.$$

and by $$(C)\ x = -\frac{35{\cdot}664}{212{\cdot}984} = -{\cdot}16744919.....$$

$$x' = +\frac{36{\cdot}331}{218{\cdot}986} = +{\cdot}16590558.....$$

It is apparent that these approximating roots may be derived from each other, an unusual circumstance in ordinary operations.

By means of the resolution of the proposed equation the following are resolved:

1. The equation $y^3 + by^2 = 1$, whose roots are $-(a^2+ab), -(b^2+ab), +ab$ when the roots of the proposed are $a+b, -a, -b$.
2. The equation $y^3 - b^3y = b^3$, since $bx = y$.
3. The equation $y^3 - ry = s$, when $\dfrac{r^3}{s^2} = b^3$, for $\dfrac{rx}{s} = y$, reference being had to the equation $x^3 - b^3x = b^3$.

Another class comprehends all cubic equations having roots real or imaginary. The same formulas are used as in the preceding resolutions, with the exception that a variable quantity is employed, which in the former was a constant, and that constant was $-b$.

Keeping to the equation $x^3 - bx = 1$, having three real roots, and where the value of b is not less than 6, let it be represented by

$$p^3 - \left(\frac{b^2}{3}n^2 + 3n + b\right)p = \left(1 - \frac{2b^3}{27}\right)n^3 + bn^2 + \frac{2b^2}{3}n + 1,$$

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where n may be any number or fraction whatever, positive or negative. Having found the approximate values of the roots of $x^3 - bx = 1$, equations are obtained in p, which are brought into relation with those in x by assuming

$$\frac{p + \frac{2bn}{3} + n^2}{np - \frac{b}{3}n^2 + 1} = x \ldots\ldots\ldots\ldots\ldots\ldots \text{(D)}.$$

Thus let the equation be $x^3 - 6x = 1$, one of whose roots has been found above, and if

$n = +1$, then we have $p^3 - 21p = 16$,
$n = -1$ .... $p^3 - 15p = -2$,
$n = +2$ .... $p^3 - 60p = -47$.
$n = -2$ .... $p^3 - 48p = 97$, *etc.*

And the roots of these equations can readily be obtained from those of the equation $x^3 - 6x = 1$. Selecting the equation $p^3 - 15p = -2$, we have by (D)

$$\frac{p-3}{-p-1} = -\cdot 16744919 = x;$$

hence

$$p = +3\cdot 8045115.$$

Means have been found to render the lower values of b effective, but these, together with a diagram, explanatory of what has been here given, will be the subject of another communication, if a page can be afforded in a subsequent number of the Mathematician.

MATHEMATICAL NOTE.

On a new method for determining the value of $\cos \frac{1}{2}A$ *in terms of the three sides of a plane triangle.*

Let ABC be a plane triangle (the figure is easily supplied), O the centre of the inscribed circle, and OD, OE, OF the radii perpendicular to the sides BC, CA, AB respectively. Denoting the sides as usual by a, b, c, and the radius of the inscribed circle by r, and putting $s = \frac{1}{2}(a+b+c)$, we get

$$\text{AE or AF} = \tfrac{1}{2}(b+c-a) = s - a,$$

and

$$\tan \tfrac{1}{2}A = \frac{\text{OE}}{\text{AE}} = \frac{r}{s-a} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, the area of the triangle ABC is expressed either by rs or $\frac{1}{2}bc \sin A$; consequently, since $\sin A = 2 \sin \frac{1}{2}A \cos \frac{1}{2}A$, we have

$$rs = \tfrac{1}{2} bc \sin A = bc \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A,$$

or

$$\frac{r}{s-a} = \frac{bc \sin \frac{1}{2} A \cos \frac{1}{2} A}{s(s-a)} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) we get

$$\tan \tfrac{1}{2} A \text{ or } \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \frac{bc \sin \frac{1}{2} A \cos \frac{1}{2} A}{s(s-a)};$$

whence

$$\cos^2 \tfrac{1}{2} A = \frac{s(s-a)}{bc}, \text{ or } \cos \tfrac{1}{2} A = \left\{\frac{s(s-a)}{bc}\right\}^{\frac{1}{2}}.$$

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MATHEMATICAL NOTE.

[*Mr. J. F. Heather, B. A., Royal Military Academy.*]

At page 66 of Walton's Collection of Mechanical Problems, I find the following paragraph.

"The friction of which we have been speaking, is the friction called into play by the rubbing of two substances against each other; the roughness of substances, however, exerts force to interrupt the production of relative motion also in the case when one body is urged to roll along another without rubbing; this may be called the friction of cohesion, depending probably upon the mutual tenacity of the two bodies. This species of friction was first noticed by Bossut, and afterwards carefully investigated by Ximenes and Coulomb: in the case of a cylinder rolling along a plane, the friction of cohesion is found to vary inversely as the diameter."

Now were the resistance here spoken of due to cohesion, then would this cohesion also offer a resistance, when any force was applied to lift the cylinder vertically off the plane; but I am not aware that any such resistance is (except between particular substances) found to exist. Moreover, such cohesion, if it did exist, could not offer any resistance to the rotatory motion of the cylinder, except under circumstances in which its assistance is no longer required to account for the observed phenomena: for if the cylinder and plane, when brought into contact, retained their perfect forms, the line of contact would have no finite breadth, and the cohesion, however great, could have no moment about this line, and could not consequently have any effect upon the rotation of the cylinder. If however we allow, as is actually the case, that the parts in contact are compressed, so that there is in fact a plane of contact of finite breadth, then the phenomena of observation are sufficiently accounted for without the assistance of cohesion.

SOLUTIONS OF MATHEMATICAL EXERCISES.

An investigation of Exercise 101 is given at page 302, by the proposer.

CII. *Mr. Fenwick.*

Find the least circumscribed ellipse to a given quadrilateral figure.

*** This problem has been solved by *Euler*, but it admits of a more concise and elegant investigation than the one given by that distinguished analyst.—*Nova Acta Acad. Petropol.* Vol. IX. p. 132.

[SOLUTION. *Mr. Thomas Weddle, Newcastle.*]

We must first find the area of the ellipse whose equation is

$$A(x-a)^2+B(y-\beta)^2+2C(x-a)(y-\beta)=D.....(a).$$

Let $2r$ denote a diameter, (x, y) one of its extremities, and ω the angle of ordination;

$$\therefore\ (x-a)^2+(y-\beta)^2+2(x-a)(y-\beta)\cos\omega=r^2.....(b).$$

Now if $2r$ be a principal diameter, (b) must be a maximum or minimum; hence differentiating (a) and (b) we have,

$$\{A(x-a)+C(y-\beta)\}dx+\{B(y-\beta)+C(x-a)\}dy=0....(c),$$

and $$\{(x-a)+\cos\omega.(y-\beta)\}dx+\{(y-\beta)+\cos\omega.(x-a)\}dy=0...(d).$$

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where n may be any number
Having found the approximate
are obtained in p, which are
assuming

$$p +$$

$$np -$$

Thus let the equation be $x^3 -$
above, and if

$n = +1$, then
$n = -1$
$n = +2$
$n = -2$

And the roots of these equati
equation $x^3 - 6x = 1$. Sele
by (D)

$$\frac{p -}{-p -}$$

hence

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rea of the ellipse is therefore, (*l*),

$$\frac{\pi \sin \omega}{LM \sin \theta} \cdot \left\{ \frac{\lambda^2+\mu^2-2\lambda\mu\cos\theta}{\sin^2\theta} + 1 \right\} \ldots\ldots\ldots (6),$$

ω is the inclination of the diagonals or axes.

ce
$$\frac{\lambda^2+\mu^2-2\lambda\mu\cos\theta}{\sin^3\theta} + \frac{1}{\sin\theta} \ldots\ldots\ldots\ldots (7),$$

ich θ is variable, must be a minimum. Differentiate (7) with respect and we have

$$\frac{2\lambda\mu}{\sin^2\theta} - \frac{3(\lambda^2+\mu^2-2\lambda\omega\cos\theta)\cos\theta}{\sin^4\theta} - \frac{\cos\theta}{\sin^3\theta} = 0.$$

ltiply by $\sin^4\theta$, put $1-\cos^2\theta$ for $\sin^2\theta$, and reduce

$$\therefore \cos^3\theta + 4\lambda\mu\cos^2\theta - (3\lambda^2+3\mu^2+1)\cos\theta + 2\lambda\mu = 0 \ldots (8).$$

ll the three roots of this cubic in $\cos\theta$ are real, but only one of them is ained between the two limits -1 and 1. This root must therefore be en as the value of $\cos\theta$, and the area, as well as the equation, of the imum ellipse, will be known from (6, 2).

r. Ambler, of Mr. Paul's school, Shooter's Hill, also favoured us with a solution. The investigations he proposer and Mr. G. W. Hearn it will be seen are given elsewhere (see p. 233 of the last, and 15 of the present number).

CIII. *By J. J.*

In integrating the partial differential equation

$$\frac{dy}{dr} = x.\frac{dy}{dx} + y,$$

x be considered a function of another variable v, and if $r+v=s$, and $r-v=s'$; prove that

$$\frac{d.xy}{d.s'} = 0.$$

[First Solution. *By J. J. the Proposer.*]

Let the proposed equation $\frac{dy}{dr} = x\frac{dy}{dx} + y$ be put under the form

$$\frac{d.xy}{dr} = x.\frac{dxy}{dx} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Then because x is a function of v,

$$x.\frac{d.xy}{dx} = x.\frac{dv}{dx}.\frac{dxy}{dv} \ldots\ldots\ldots\ldots\ldots (2).$$

Suppose $\frac{dv}{dx} = \frac{1}{x}$; then from (1) and (2) we get

$$\frac{d.xy}{dr} = \frac{dxy}{dv} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

By hypothesis, $r+v=s$, and $r-v=s'$, consequently xy is a function of s and s': therefore

$$\frac{d.xy}{dr} = \frac{d.xy}{ds}.\frac{ds}{dr} + \frac{dxy}{ds'}.\frac{ds'}{dr} = \frac{d.xy}{ds} + \frac{dxy}{ds'}.$$

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Multiply (c) by λ and (d) by μ, subtract and equate the coefficients of dx and dy to zero,

$$\therefore \{A(x-a)+C(y-\beta)\}\lambda=\{(x-a)+\cos\omega.(y-\beta)\}\mu \ldots\ldots(e),$$

and $$\{B(y-\beta)+C(x-a)\}\lambda=\{(y-\beta)+\cos\omega.(x-a)\}\mu \ldots\ldots(f).$$

Multiply (e) by $(x-a)$, and f by $(y-\beta)$, add and we have, by (a, b), $D\lambda=r^2\mu$, and we may therefore take $\lambda=r^2$ and $\mu=D$; hence (e) and (f) become

$$(Ar^2-1)(x-a)=(D\cos\omega-Cr^2)(y-\beta)\ldots\ldots(g),$$
$$(Br^2-D)(y-\beta)=(D\cos\omega-Cr^2)(x-a)\ldots\ldots(h).$$

Multiply (g) and (h), cancel the common factors, and arrange according to powers of r^2,

$$\therefore (AB-C^2)r^4-(A+B-2C\cos\omega)Dr^3+D^2\sin^2\omega=0\ldots.(i).$$

Now if m and n be the principal semiaxes, it is evident that m^2 and n^2 are the roots of (i), and hence by the theory of equations

$$m^2n^2=\frac{D^2\sin^2\omega}{AB-C^2} \ldots\ldots\ldots\ldots\ldots\ldots(k).$$

Moreover the area of the ellipse $=\pi mn$;

$$\therefore \text{area of the ellipse}=\frac{\pi D\sin\omega}{\{AB-C^2\}^{\frac{1}{2}}} \ldots\ldots\ldots\ldots\ldots\ldots(l),$$

the formula required, which is very simple.

We are now enabled to proceed to the solution of the exercise.

Taking the diagonals of the quadrilateral for axes, its angular points may be denoted by $(-a, 0)$, $(0, b)$, $(c, 0)$ and $(0, -d)$, and we easily find the equation to the circumscribed ellipse to be

$$\frac{1}{ac}x^2+\frac{1}{bd}y^2+2kxy+\left(\frac{1}{c}-\frac{1}{a}\right)x+\left(\frac{1}{b}-\frac{1}{d}\right)y-1=0\ldots.(1),$$

where k is arbitrary. Assume $L^2=\frac{1}{ac}$, $M^2=\frac{1}{bc}$, $2\lambda=\left\{\frac{a}{c}\right\}^{\frac{1}{2}}-\left\{\frac{c}{a}\right\}^{\frac{1}{2}}$, and $2\mu=\left\{\frac{d}{b}\right\}^{\frac{1}{2}}-\left\{\frac{b}{d}\right\}^{\frac{1}{2}}$; also let $k=LM\cos\theta$, this assumption being allowable since (1) denotes an ellipse. Hence (1) becomes

$$L^2x^2+M^2y^2+2LM\cos\theta.xy+2L\lambda x+2M\mu y-1=0\ldots\ldots(2).$$

Let (α, β) denote the centre of (2) then we have

$$\left.\begin{aligned} L\alpha+M\cos\theta\beta+\lambda&=0\\ M\beta+L\cos\theta\alpha+\mu&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots(3),$$

whence,
$$L\alpha=\frac{\mu\cos\theta-\lambda}{\sin^2\theta},\quad M\beta=\frac{\lambda\cos\theta-\mu}{\sin^2\theta}\ldots\ldots\ldots\ldots(4).$$

By means of (3) and (4) the equation (2) easily becomes

$$L^2(x-\alpha)^2+M^2(y-\beta)^2+2LM\cos\theta\,(x-\alpha)(y-\beta)=\frac{\lambda^2+\mu^2-2\lambda\mu\cos\theta}{\sin^2\theta}+1$$
$$\ldots\ldots(5).$$

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The area of the ellipse is therefore, (l),

$$\frac{\pi \sin \omega}{\mathrm{LM} \sin \theta} \cdot \left\{ \frac{\lambda^2 + \mu^2 - 2\lambda\mu \cos \theta}{\sin^2 \theta} + 1 \right\} \ldots\ldots\ldots (6),$$

where ω is the inclination of the diagonals or axes.

Hence $$\frac{\lambda^2 + \mu^2 - 2\lambda\mu \cos \theta}{\sin^3 \theta} + \frac{1}{\sin \theta} \ldots\ldots\ldots\ldots (7),$$

in which θ is variable, must be a minimum. Differentiate (7) with respect to θ, and we have

$$\frac{2\lambda\mu}{\sin^2\theta} - \frac{3(\lambda^2 + \mu^2 - 2\lambda\omega \cos \theta)\cos \theta}{\sin^4 \theta} - \frac{\cos \theta}{\sin^2\theta} = 0.$$

Multiply by $\sin^4 \theta$, put $1 - \cos^2 \theta$ for $\sin^2 \theta$, and reduce

$$\therefore \cos^3 \theta + 4\lambda\mu \cos^2 \theta - (3\lambda^2 + 3\mu^2 + 1)\cos \theta + 2\lambda\mu = 0 \ldots . (8).$$

All the three roots of this cubic in $\cos \theta$ are real, but only one of them is contained between the two limits -1 and 1. This root must therefore be taken as the value of $\cos \theta$, and the area, as well as the equation, of the minimum ellipse, will be known from (6, 2).

Mr. Ambler, of Mr. Paul's school, Shooter's Hill, also favoured us with a solution. The investigations of the proposer and Mr. G. W. Hearn it will be seen are given elsewhere (see p. 233 of the last, and p. 315 of the present number).

CIII. *By J. J.*

In integrating the partial differential equation

$$\frac{dy}{dr} = x \cdot \frac{dy}{dx} + y,$$

if x be considered a function of another variable v, and if $r + v = s$, and $r - v = s'$; prove that

$$\frac{d.xy}{d.s'} = 0.$$

[First Solution. *By J. J. the Proposer.*]

Let the proposed equation $\frac{dy}{dr} = x \frac{dy}{dx} + y$ be put under the form

$$\frac{d.xy}{dr} = x \cdot \frac{dxy}{dx} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Then because x is a function of v,

$$x \cdot \frac{d.xy}{dx} = x \cdot \frac{dv}{dx} \cdot \frac{dxy}{dv} \ldots\ldots\ldots\ldots\ldots (2).$$

Suppose $\frac{dv}{dx} = \frac{1}{x}$; then from (1) and (2) we get

$$\frac{d.xy}{dr} = \frac{dxy}{dv} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

By hypothesis, $r + v = s$, and $r - v = s'$, consequently xy is a function of s and s': therefore

$$\frac{d.xy}{dr} = \frac{d.xy}{ds} \cdot \frac{ds}{dr} + \frac{dxy}{ds'} \cdot \frac{ds'}{dr} = \frac{d.xy}{ds} + \frac{dxy}{ds'}.$$

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Similarly, $$\frac{dxy}{dv} = \frac{dxy}{ds} - \frac{dxy}{ds'};$$

hence from (3) it is obvious that $\frac{dxy}{ds'} = 0$, which is the proof required.

The integration of the proposed equation may be completed in the following manner:

The expression $\frac{d.xy}{ds'} = 0$, shews that xy is independent of s', and is satisfied by any function of s. Put therefore $xy = \mathrm{F}(s) = \mathrm{F}(r+v)$, then the above assumption $\frac{dv}{dx} = \frac{1}{x}$, gives $v = \log x$; therefore

$$xy = \mathrm{F}(r+\log x) = \mathrm{F}\ (\log c^r x) = f(xc^r):$$

$$\therefore\ y = \frac{1}{x} f\ (x.c^r) = c^r \frac{1}{xc^r} f(x.c^r) = c^r \phi(xc^r).$$

See LAPLACE's Theorie Analytique des Probabilities, p. 285; and the late ingenious Mr. THOMAS's Notes on that work.

[SECOND SOLUTION. *Mr. Thomas Weddle.*]

Here $$\frac{dy}{dr} = \frac{dy}{ds}.\frac{ds}{dr} + \frac{dy}{ds'}.\frac{ds'}{dr} = \frac{dy}{ds} + \frac{dy}{ds'},$$

and $$\frac{dy}{dx} = \frac{dy}{ds}.\frac{ds}{dx} + \frac{dy}{ds'}.\frac{ds'}{dx} = \left(\frac{dy}{ds} - \frac{dy}{ds'}\right)\frac{dv}{dx}.$$

Hence the equation becomes

$$\frac{dy}{ds} + \frac{dy}{ds'} = x.\left\{\frac{dy}{ds} - \frac{dy}{ds'}\right\}\frac{dv}{dx} + y \quad \ldots\ldots\ldots\ldots (1)$$

Assume $$\frac{dy}{ds} = x\,\frac{dy}{ds}.\frac{dv}{dx},\ \text{or}\ \frac{dv}{dx} = \frac{1}{x};$$

$$\therefore\ v = \log x \quad \ldots\ldots\ldots\ldots\ (2).$$

Hence (1) reduces to

$$\frac{dy}{ds'} = -\frac{dy}{ds'} + y,\ \text{or}\ \frac{dy}{ds'} = \tfrac{1}{2}y \quad \ldots\ldots\ldots\ldots\ldots (3).$$

Also since v or $\log x = \frac{1}{2}(s - s')$, we have

$$\frac{dx}{ds'} = -\frac{1}{2}x \quad \ldots\ldots\ldots\ldots\ldots\ (4).$$

Hence $$\frac{d.xy}{ds'} = x\,\frac{dy}{ds'} + y\,\frac{dx}{ds'} = (3, 4)\tfrac{1}{2}xy - \tfrac{1}{2}xy = 0.$$

Cor. Hence, integrating,

$$xy = \text{arbitrary function of } s = \phi s \text{ (suppose)};$$

$$\therefore\ xy = \phi\ (r + \log x)$$

is the solution of the given partial differential equation, the form of ϕ being arbitrary.

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CIV. *Mr. G. W. Hearn, R. M. College.*

Find the moments of inertia of the five regular solids about any axis through their centres of gravity.

[FIRST SOLUTION. *Mr. G. W. Hearn, the Proposer.*]

Lemma. Let A, B, C be the moments of inertia of any of the five regular solids about principal axes through the centre; then A, B, C are all equal, and consequently the moments about all other axes through the centre are also equal.

For wherever the principal axes be situated, it is evidently possible to find other axes similarly situated with respect to the figure about which the moments are the same as A, B, or C, even supposing A, B, and C to be all different. If possible therefore let $A > B$ and $B > C$, and let α, β, λ be the angles of direction of another axis, about which the moment is C. Then by the usual formulæ

$$A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma = C,$$
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$
$$\therefore (A - C)\cos^2\alpha + (B - C)\cos^2\beta = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But since by hypothesis $A > C$ and $B > C$; $A - C$ and $B - C$ are each positive, and equation (1) cannot have place. Hence the hypothesis of the inequality of A, B, and C leads to an absurdity. Therefore A, B, and C are not unequal, that is, they are equal.

PROP. I. To find the moment of inertia of a right pyramid whose base is any plane regular figure, about an axis inclined at an angle α to the axis of the pyramid, and passing through its vertex.

Take a section at the distance x from the vertex parallel to the base and suppose the pyramid to consist of laminæ of thickness dx. The square of the radius of gyration of the lamina at the distance x may be represented by ex^2, e being a constant, and the mass of the lamina $= \mu x^2 dx$, μ being another constant. Hence the moment of inertia of the lamina about the axis of the pyramid $= e\mu x^4 dx$. Also its moment of inertia about an axis in its own plane through the centre $= \frac{1}{4}e\mu x^4 dx$, by the well known dynamical formulæ applicable to plane figures.

Suppose a plane to pass through the axis of the pyramid and the axis through the vertex about which the moment is required, this plane will cut the lamina in a line through its centre, and the moment of inertia of the lamina about this line as an axis will be $\frac{1}{8}e\mu x^4 dx$; and hence the moment about an axis parallel to this last and passing through the vertex of the pyramid is

$$\mu x^2 dx . x^2 + \tfrac{1}{4}e\mu x^4 dx = \mu x^4 dx(1 + \tfrac{1}{8}e).$$

Hence by the usual formula, the moment of inertia of the lamina about the given axis through the vertex

$$= e\mu x^4 dx . \cos^2\alpha + \mu x^4 dx(1 + \tfrac{1}{8}e)\sin^2\alpha$$
$$= \mu x^4 dx\{e\cos^2\alpha + (1 + \tfrac{1}{8}e)\sin^2\alpha\}$$
$$= \mu x^4 dx\{(1 + \tfrac{1}{8}e) - (1 - \tfrac{1}{8}e)\cos^2\alpha\}.$$

Integrating, with respect to x, from $x = 0$ to $x = a$, where a is the axis of the pyramid, we have for the moment of inertia of the pyramid about the same axis

$$\tfrac{1}{5}\mu a^5 \{(1 + \tfrac{1}{8}e) - (1 - \tfrac{1}{8}e)\cos^2\alpha\};$$

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but $\int_0^a \mu x^2 dx = \frac{1}{3}\mu a^3 =$ mass of pyramid $=$ P suppose;

$$\therefore \quad \tfrac{3}{5} Pa^2\{(1+\tfrac{1}{2}e)-(1-\tfrac{1}{2}e)\cos^2 a\},$$

is the same moment of inertia.

PROP. II. To deduce from the former the moment of inertia of any of the five regular solids.

These figures may all be divided into pyramids such as in the preceding proposition, having their common vertex in the centre of the figure, and their several bases the several plane faces of the figure. The axes of such pyramids will make different angles with an axis through the centre of which we consider a as the general representative.

Then since there are as many pyramidal axes as plane faces, if n be their number, we have for the sum of the moments of the pyramids, or moment of the regular figure of n plane faces,

$$\tfrac{3}{5} Pa^2\{n(1+\tfrac{1}{2}e)-(1-\tfrac{1}{2}e)\Sigma\cos^2 a\};$$

but by the *lemma* this moment is independent of a, and therefore $\Sigma\cos^2 a$ is independent of a. This circumstance enables us to assume an axis through the centre most convenient for the calculation of $\Sigma\cos^2 a$ in each case. Now the finding of $\Sigma\cos^2 a$ is a purely spherical problem which may be stated thus:

Let any of the five regular solids be inscribed in a sphere, and let any other point be chosen on the sphere, and let α, β, γ, *etc.* be the angular distances of the assumed point from the several angular points of the figure, it is required to find the sum

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + etc, \text{ or } \Sigma\cos^2 a.$$

I find that if n be the number of angular points, we have $\Sigma\cos^2 a = \frac{n}{3}$, but as I believe the proposition to be entirely new, I do not here give the demonstration of it, intending to offer it to the consideration of your correspondents. If it is not new, probably a demonstration already exists.

The five regular solids are connected in the following simple manner.

If perpendiculars be drawn to the plane faces of a tetrahedron they will intersect the surface of a concentric sphere in points which are the vertices of another tetrahedron.

If a hexahedron be so treated an octahedron will be formed, and conversely.

If a dodecahedron be so treated an icosahedron will be formed, and conversely.

In all cases then the number of points or terms in $\Sigma\cos^2 a$ will be the same as the number of pyramidal axes, or the same as the number of plane faces, and therefore $= n$. Putting $\Sigma\cos^2 a = \frac{n}{3}$ in the formula, it becomes

$$\tfrac{1}{5} Pna^2\{3(1+\tfrac{1}{2}e)-(1-\tfrac{1}{2}e)\}$$

$$= \tfrac{2}{5} Ma^2(1+e), \text{ where } M = nP.$$

That is, the moment of inertia of any of the five regular solids about an axis through its centre is represented by

$$\tfrac{2}{5} Ma^2(1+e),$$

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where M is its mass, a the radius of the inscribed sphere, and $e = \frac{k^2}{a^2}$, where k is the radius of gyration of one of its plane faces about an axis through its centre perpendicular to its plane.

I will not trouble you with an investigation of the numerical value of e for all the five solids, as this is easily accomplished, and shall content myself with the following table of their values.

Figure.	Value of e.
1 Tetrahedron	2
2 Hexahedron	$\frac{2}{3}$
3 Octahedron	$\frac{1}{2}$
4 Dodecahedron	$\frac{1}{2}\left(1-\frac{2}{3}\sin^2\frac{\pi}{5}\right)\left(3\tan^2\frac{\pi}{5}-1\right)$
5 Icosahedron	$\frac{1}{4}\left(3\tan^2\frac{\pi}{5}-1\right)$.

I will merely remark, that these values have been calculated from the formulæ

$$\cos\Psi = \tan\left(1-\frac{2}{n}\right)\frac{\pi}{m}\cot\frac{\pi}{m},$$

$$\frac{c}{a} = \tan\Psi,$$

where n is the number of plane faces, m the number of sides of each face, and c the radius of the circumscribing circle of each plane face, a as before denoting the radius of the inscribed sphere; and then in the 1st, 3rd, and 5th solids $k^2 = \frac{1}{4}c^2$; in the second, $k^2 = \frac{1}{3}c^2$; and in the fourth, $k^2 = \frac{1}{3}c^2\left(1-\frac{2}{3}\sin^2\frac{\pi}{5}\right)$, all of which formulæ are elementary.

Note. This exercise was suggested by Mr. Elliott's problem of finding the moments of inertia of any plane regular figure, and in this it might have been shewn à priori that the moments of inertia round all axes through the centre in the plane of the figure are the same, in a similar way to that employed for the solids. For instance, in the hexagon ABCDEF, let CF be the line joining two opposite angular points, MN and PQ two other lines each bisecting two opposite sides of the hexagon; then CF and MN are evidently principal axes. Let the moment about MN = A, and moment about FC = B. But the moment about PQ is the same as that about MN, and therefore = A; and $\sphericalangle$ MOP $= \frac{\pi}{3}$;

$$\therefore \quad A\cos^2\frac{\pi}{3} + B\sin^2\frac{\pi}{3} = A,$$

$$\text{or} \quad B\sin^2\frac{\pi}{3} = A\left(1-\cos^2\frac{\pi}{3}\right) = A\sin^2\frac{\pi}{3};$$

$$\therefore \quad B = A.$$

[Second Solution. *Mr. Thomas Weddle, Newcastle.*]

Let r be the radius of the sphere inscribed in the body and a that of the circle circumscribing each plane face; also m = number of plane angles forming each solid angle, n = number of sides of each face, p = number of

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faces, i = inclination of two contiguous faces, and ρ = density of the body. By *Hind's Trigonometry*, pp. 219, 220, we have,

$$\sin\frac{1}{2}i = \cos\frac{\pi}{m} \div \sin\frac{\pi}{n} \ldots\ldots(1), \quad r = a\cos\frac{\pi}{n}.\tan\frac{1}{2}i \ldots\ldots\ldots\ldots(2),$$

$$\text{Area of each face} = f = \frac{1}{2}n a^2 \sin\frac{2\pi}{n} \ldots\ldots\ldots\ldots(3);$$

$$\text{Volume of the body} = V = \frac{1}{6}n p a^2 r \sin\frac{2\pi}{n} \ldots\ldots\ldots\ldots(4).$$

Take the centre O of the body for the origin of rectangular axes Ox, Oy, Oz, draw OP perpendicular to F one of the faces, meeting it in P, and let α, β, γ be the directing cosines of OP. Since α is the sine of the angle which the plane of F makes with (PQ) a parallel to Ox through P, the moment of inertia of F about PQ is ("Mathematician," vol. II. p. 114),

$$\frac{1}{24}n\rho a^4 \sin\frac{2\pi}{n}\left\{2+\cos\frac{\pi}{n}\right\}\left\{1+a^2\right\}.$$

Now the mass of F is, (3), $\frac{1}{2}n\rho a^2 \sin\frac{2\pi}{n}$, and the square of the distance of P (which is the centre of gravity of F) from Ox is $r^2(1-\alpha^2)$; hence by a well-known theorem (see *Poisson's Mécanique*, tome II. p. 53) the moment of inertia of F about Ox is

$$\frac{1}{24}n\rho a^4 \sin\frac{2\pi}{n}.\left\{2+\cos\frac{2\pi}{n}\right\}.\left\{1+a^2\right\}+\frac{1}{2}n\rho a^2 r^2 \sin\frac{2\pi}{n}(1-\alpha^2)\ldots(5).$$

Let A, B, C be the moments of inertia of the surface of the regular body about the axes of x, y and z respectively; hence summing (5) for all the p faces, we have

$$A=\frac{1}{24}n\rho a^4 \sin\frac{2\pi}{n}.\left\{2+\cos\frac{2\pi}{n}\right\}.\left\{p+\Sigma\alpha^2\right\}+\frac{1}{2}n\rho a^2 r^2 \sin\frac{2\pi}{n}.\left\{p-\Sigma\alpha^2\right\} \ldots\ldots(6).$$

Similarly,

$$B=\frac{1}{24}n\rho a^4 \sin\frac{2\pi}{n}.\left\{2+\cos\frac{2\pi}{n}\right\}.\left\{p+\Sigma\beta^2\right\}+\frac{1}{2}n\rho a^2 r^2 \sin\frac{2\pi}{n}.\left\{p-\Sigma\beta^2\right\} \ldots\ldots(7);$$

and,

$$C=\frac{1}{24}n\rho a^4 \sin\frac{2\pi}{n}.\left\{2+\cos\frac{2\pi}{n}\right\}.\left\{p+\Sigma\gamma^2\right\}+\frac{1}{2}n\rho a^2 r^2 \sin\frac{2\pi}{n}.\left\{p-\Sigma\gamma^2\right\} \ldots\ldots(8).$$

Moreover it is easily seen that the moments of inertia of the surface of the regular body about all axes through O are equal. In fact, if the moment of inertia be taken about any axis through O, we can assign other axes about which the moments must, from the symmetry of the surface, be the same. Hence the axis of both the greatest and the least moment must admit of multiple positions, which we know can only be the case when the three principal moments are equal, and then the moments about *all* axes through O will be equal to each other. Hence A = B = C, and if (6, 7, 8) be added, recollecting that $\Sigma\alpha^2+\Sigma\beta^2+\Sigma\gamma^2 = \Sigma(\alpha^2+\beta^2+\gamma^2)$ = number of faces = p, we shall get

$$A = B = C = \frac{1}{18}n p \rho a^4 \sin\frac{2\pi}{n}.\left\{2+\cos\frac{2\pi}{n}\right\} + \frac{1}{3}n p \rho a^2 r^2 \sin\frac{2\pi}{n} \ldots\ldots\ldots\ldots(9),$$

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which is the moment of inertia of the surface of the body about any axis through its centre of gravity.

Suppose now that in the given body two other bodies are described, concentric with and similar to it, and such that the radii of their inscribed spheres may be r' and $r' + \delta r'$ respectively; then supposing the faces of all the bodies to be parallel, the solid comprised between the surfaces of the two bodies may be taken as the element of the given solid. Now the moment of this element about any axis through O is, (9),

$$\frac{1}{18} n p \rho a'^4 . \delta r' \sin\frac{2\pi}{n} \left\{2 + \cos\frac{2\pi}{n}\right\} + \frac{1}{3} n p \rho a'^2 r'^2 \delta r' \sin\frac{2\pi}{n} \quad \ldots\ldots (10),$$

in which a' is the radius of the circle circumscribing one of the smaller of the two bodies. Moreover since, (2), $r' = a' \cos\frac{\pi}{n} \tan\frac{1}{2} i$, it is easy to see that $\int a'^4 . dr' = \frac{1}{5} a' r'$ and $\int a'^2 r'^2 dr' = \frac{1}{5} a'^2 r'^3$; hence integrating (10), between the limits $r' = 0$, and $r' = r$, we have,

$$M k^2 = \frac{1}{90} . n p \rho a^4 r \sin\frac{2\pi}{n} \left\{2 + \cos\frac{2\pi}{n}\right\} + \frac{1}{15} n p \rho a^2 r^3 \sin\frac{2\pi}{n} \quad \ldots\ldots (11).$$

This is the moment of inertia of the regular body about any axis through its centre of gravity, and it is consequently the expression required.

Moreover $M = \rho V = \frac{1}{6} n p \rho a^2 r \sin\frac{2\pi}{n}$, by (4); hence, dividing (11) by this, we have,

$$k^2 = \frac{1}{15} a^2 \left\{2 + \cos\frac{2\pi}{n}\right\} + \frac{2}{5} r^2 \ldots\ldots\ldots\ldots\ldots (12),$$

which is very neat.

One of the quantities a, r might be eliminated from (11) and (12) by means of (2), but these equations are simpler as they are.

For the hexahedron or cube (12) becomes, $k^2 = \frac{2}{3} r^2$ (a known result), for the tetrahedron, octahedron, and icosahedron, $k^2 = \frac{1}{10} a^2 + \frac{2}{5} r^2$, and for the dodecahedron, $k^2 = \frac{7 + \sqrt{5}}{60} a^2 + \frac{2}{5} r^2$.

CV. *Mr. G. W. Hearn.*

A cavity is scooped out of a solid sphere of given substance, and all that is known about it is, that it is one of the regular solids concentric with the sphere. It is required to determine by mechanical means (without opening or penetrating the sphere) the form and dimensions of the cavity.

[First Solution. *Mr. G. W. Hearn, the proposer.*]

1st. Weigh the sphere, and let w = its weight. Since we know the substance of the sphere we can find what would have been its weight had had it been solid; let this = W. Then $W - w$ = weight of the portion scooped out, $\frac{W - w}{g} = M$, the mass of the unknown regular solid, which is therefore given.

2nd. Supposing a = the radius of inscribed sphere of the figure, and n the number of its plane faces, we have an equation between n and a of the from

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$$M = \rho v_n a^3,$$

ρ being the density of the substance, and v_n a function of n easily found for each solid.

3rd. Suspend the hollow sphere and let it vibrate in a vertical plane, and let t seconds be the time of vibration determined by counting the number of vibrations in a given time. Also let h be the distance of the axis of suspension from the centre.

Let $$I_n = \frac{2}{5} Ma^2 \left(1 + e_n\right),$$

the moment of inertia of the unknown figure about an axis through its centre. Let R be the given or ascertainable radius of the sphere, then $\frac{2}{5} \cdot \frac{W}{g} R^2$ = moment of inertia of the sphere, supposed solid, about an axis through its centre. Hence moment of inertia of the shell about an axis through the centre is

$$\frac{2}{5} \cdot \frac{W}{g} \cdot R^2 - I_n;$$

and its moment about axis of suspension

$$= \left(\frac{W}{g} - M\right) h^2 + \frac{2}{5} \cdot \frac{W}{g} \cdot R^2 - I_n$$

$$= \frac{w}{g} \cdot h^2 + \frac{2}{5} \cdot \frac{W}{g} \cdot R^2 - \frac{2}{5} Ma^2 (1 + e_n)$$

$$= \mu, \text{ suppose.}$$

And the equation of motion is as usual

$$\mu \frac{d^2\theta}{dt^2} + \frac{w}{g} \cdot h\theta = 0,$$

and

$$t = 2\pi \div \left\{\frac{wh}{\mu}\right\}^{\frac{1}{2}} = 2\pi \left\{\frac{\mu}{wh}\right\}^{\frac{1}{2}},$$

$$\therefore \quad \mu = wh \cdot \frac{t^2}{4\pi^2}.$$

Hence $$\frac{w}{g} \cdot h^2 + \frac{2}{5} \cdot \frac{W}{g} \cdot R^2 - \frac{2}{5} Ma^2 (1 + e_n) = wh \cdot \frac{t^2}{4\pi^2},$$

or $$\frac{2}{5} Ma^2 (1 + e_n) = \frac{w}{g} h^2 + \frac{2}{5} \cdot \frac{W}{g} \cdot R^2 - wh \cdot \frac{t^2}{4\pi^2};$$

but $$a^2 = \left(\frac{M}{\rho}\right)^{\frac{2}{3}} \left(\frac{1}{v_n}\right)^{\frac{2}{3}},$$

$$\therefore \quad \frac{1 + e_n}{v_n^{\frac{1}{3}}} = \frac{\frac{5}{2} \rho^{\frac{2}{3}} \left\{\frac{w}{g} \cdot h^2 + \frac{2}{5} \cdot \frac{W}{g} \cdot R^2 - wh \cdot \frac{t^2}{4\pi^2}\right\}}{M^{\frac{2}{3}}}.$$

Now e_n and v_n are known numerical functions of n, e_n having been given for each solid in the preceding problem, and v_n = volume of a regular solid of n faces, the radius of inscribed sphere being unity, which is well known. Hence of the five possible values which the function $\frac{1 + e_n}{v_n^{\frac{1}{3}}}$ is capable of having for the values $n = 4, 6, 8, 12, 20$, that which most nearly satisfies the above formula will determine the solid.

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[SECOND SOLUTION. *Mr. Thomas Weddle; and similarly by Dr. Burns.*]

Ascertain the weight W and the radius R of the given sphere, and let its known density be ρ. Had there been no cavity, the weight of the sphere would have been $\frac{4}{3}\pi g\rho R^3$, hence if V be the volume of the cavity, we shall have

$$g\rho V = \frac{4}{3}\pi g\rho R^3 - W,$$

$$\text{or} \quad V = \frac{4}{3}\pi R^3 - \frac{W}{g\rho} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

Also by equations (2, 4) of my solution to the last question, we have

$$V = \frac{1}{6} npa^3 \sin\frac{2\pi}{n} \cos\frac{\pi}{n} \tan\tfrac{1}{2} i \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

Suspend the sphere by a slender string, the known length of which (including the radius of the sphere) denote by l; cause the sphere to make small oscillations, and observe the number p of oscillations in a given time t; hence if k' denote the radius of gyration of the sphere about a horizontal axis through its centre, we shall have

$$\frac{p}{t} = \rho \left\{\frac{l^2 + k'^2}{lg}\right\}^{\frac{1}{2}}, \text{ or } k'^2 = \frac{glp^2}{\pi^2 t^2} - l^2 \ldots\ldots\ldots\ldots (c).$$

Moreover had there been no cavity, the moment of inertia of the sphere about any axis through its centre would have been $\frac{8}{15}\pi\rho R^5$, and that of the portion now hollow, $\rho V k^2$, (k having the same value as in (12), see solution to the last exercise);

$$\therefore \frac{W}{g} k'^2 = \frac{8}{15}\pi\rho R^5 - \rho V k^2,$$

$$\text{or} \quad \frac{W}{g\rho} k'^2 = \frac{8}{15}\pi R^5 - V k^2 \ldots\ldots\ldots\ldots\ldots\ldots (d).$$

Having obtained these equations, we may now proceed as follows:— Calculate V, a, r, k' and k by means of (a), (b), (2), (c), and (12), on the supposition that the cavity is of the form of one of the regular bodies (the tetrahedron, for instance), and find whether (d) is satisfied by these values; if it is, this is the form of the cavity, if not, we must calculate for another of the regular bodies, and so on, until we find (d) to be satisfied. When the form of the cavity has been thus ascertained, the previously computed value of a or r will determine its dimensions.

CVI. *Mr. Davies.*

A complete spherical quadrilateral and its three diagonals are formed of great circles, and through the extremities of each diagonal and the intersections of the other two, other great circles are drawn: it is required to prove, independently of projective considerations, that all the circles concerned in the figure are so divided by each other, that the sines of their segments are in harmonical proportion.

[*Demonstration by the Proposer.*]

1. *Definition of the figure.* Let O be the centre of the sphere upon which the simple salient quadrilateral ABCD is described; all the circles

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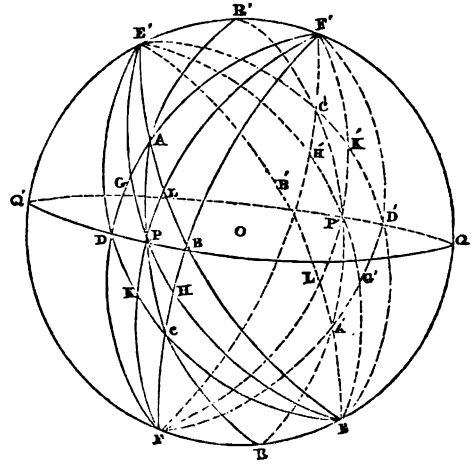
here spoken of being understood as *great* circles of the sphere. Let E and F be the intersections of the opposite sides of the quadrilateral; viz. of AB, CD and of BC, AD: and draw AC, BD, EF intersecting respectively two and two in P, Q, R as in the figure. Also, draw the circles EP, FP meeting the sides of the quadrilateral ABCD in G, H, K, L; and complete the several great circles now specified, upon the surface of the sphere.

Now for *every intersection* of two circles of this figure (as A, for instance) there will be a second (A′) made by the intersection of the same two circles AB, AD diametrically opposite to it on the spherical surface. For the purpose of more readily distinguishing the parts of the figure by the eye, the parts on the two hemispheres separated by the circle EFE′F′ are marked in plain and in dotted line; and every point diametrically opposite to a point already described is marked by the same letter accented.



Then the entire figure formed by the four circles AB, BC, CD, DA being completed so as to intersect each other mutually in E, E′, F′, F, A′, B′, C′, D′, A, B, C, D constitutes the *complete spherical quadrilateral;* and the three great circles AC, BD, EF being completed, constitute its three diagonals. The circles EP, FP are specified in the enunciation of the property; but they have received no special name.

2. *Preliminary properties of the figure.* Reference might be made to Carnot's *Essai sur les Transversales**, to a paper of my own in Leybourn's Repository, vol. vi.†, or to the second volume of my edition of Hutton's Course‡, for some of these: but to avoid the scattered reference to works which might not be in the hands of every reader, it is deemed advisable to give them here, so as to render the entire investigation complete from beginning to end.

(*a.*) *If a spherical triangle* ABD *be cut by a transversal in* E, F, Q, *we shall have the following relation amongst its segments:—*

$$\sin AF \sin BE \sin DQ = \sin AE \sin BQ \sin DF.$$

For by a fundamental property of the spherical triangles AFE, BEQ, FQD, we have

$$\sin AF : \sin AE :: \sin E : \sin F,$$
$$\sin BE : \sin BQ :: \sin Q : \sin E,$$
$$\sin DQ : \sin DF :: \sin F : \sin Q;$$

and multiplying, *etc.*, we get the equation as above stated.

In the figure, the three sides are cut in their extensions in E, F, Q (or E′, F′, Q′): but the demonstration is equally applicable if only one of the

* Page 86. † Page 67. ‡ Page 78.

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sides be cut in its extension. This will be seen at once by constructing the figure to correspond to the case; or, which is nearly the same thing, by supposing the triangle EAF to be cut by the transversal in B, D, Q, and then forming the requisite proportions. It will, moreover, in some cases, be necessary to substitute the sine of the supplement for the sine of the angle or the arc concerned; which being *always of the same value* is perfectly allowable in a research which relates to *values* only.

(*b*) *If through any point* C *on the spherical surface and the angular points* A, B, D, *of a spherical triangle, transversals be drawn to cut the sides* AB, BD, DA *(produced or not as the position of* C *may require) in* P, E, F; *then the following relations amongst the segments will subsist:—*

$$\sin AF \sin BE \sin DP = \sin AE \sin BP \sin DF.$$

For the triangle ABD is divided into two others ABP, APD by the transversal AC: and the triangle ABP is cut by the transversal DE in D, C, E; and APD by BF in B, C, F. Whence, by the preceding property, we have

$$\sin BD \sin PC \sin AE = \sin DP \sin CA \sin EB,$$
$$\sin BP \sin CA \sin FD = \sin DB \sin PC \sin AF;$$

and multiplying these we find the stated property to result.

The same remarks as to the extensions of the sides of the triangles, and the substitution of supplements for the arcs themselves, apply here as in the preceding case. Carnot, however, limits the point C to be *within* the field of the triangle ABD, which is an unnecessary and inconvenient restriction*. The method of investigation is, however, entirely due to that distinguished geometer.

(*c*.) *Let an arc* AB *of a great circle be divided in* C *and* D *so that*

$$\sin AC : \sin CB :: \sin AD : \sin DB;$$

also let arcs be drawn from A, B, C, D *to any point* P *on the sphere, and these be cut by another spherical transversal in* A′, B′, C′, D′: *then,*

$$\sin A'C' : \sin C'B' :: \sin A'D' : D'B'.$$

(The reader will readily sketch the figure.)

For,

$$\sin AC : \sin CP :: \sin APC : \sin PAC = \sin PAD,$$
$$\sin BD : \sin DP :: \sin BPD : \sin PBD = \sin PBC,$$
$$\sin CP : \sin CB :: \sin PBC : \sin CPB,$$
$$\sin DP : \sin DA :: \sin PAD : \sin APD.$$

Compounding which, we obtain

$$\sin AC \sin BD : \sin CB \sin AD :: \sin APC \sin BPD : \sin CPB \sin APD;$$

or, taking account of the hypothesis, this becomes

$$\sin APC \sin BPD = \sin CPB \sin APD.$$

* It is singular that Carnot (having conceived, and so efficiently used, the idea of the *complete quadrilateral in plano)* should not have considered the *complete spherical triangle:* viz. that figure which is formed of the entire circles, the three segments of which (one segment of each circle) form the ordinary objects of spherical trigonometry. No general researches into the properties of spherical triangles can be complete without taking into consideration the whole system of these circles; or what I have ventured elsewhere to call the "associated system" and its symmetrical system. With Carnot, however, the properties of spherical figures were only objects of investigation *incidentally*; and he does not appear to have attempted any researches respecting them except so far as some *direct* analogy with plane triangles might have suggested any spherical property to his mind.

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By the nature of the inquiry, also, the hypothesis and conclusion are convertible; hence the converse is also true: viz. that if

$$\sin APC : \sin CPB :: \sin APD : \sin BPD,$$

then also,

$$\sin AC : \sin CB :: \sin AD : \sin DB,$$

wherever the transversal AB may be taken so as to cut the four arcs AP, BP, CP, DP; and hence when they are taken in the position A'B'. Whence the property is true.

3. *Demonstration of the properties of the complete quadrilateral.* From (a) and (b) we have

$$\sin DQ : \sin QB :: \sin AE \sin DF : \sin EB \sin FA,$$

$$\sin DP : \sin PB :: \sin AE \sin DF : \sin EB \sin FA;$$

whence it follows that

$$\sin DQ : \sin QB :: \sin DP : \sin PB.$$

That is, the diagonal DB of the spherical quadrilateral ABCD is harmonically divided in P and Q.

In the same way it is shewn that the diagonal AC is harmonically divided in P and R, and FE in R and Q.

Again, since DB is harmonically divided in P and Q, and arcs are drawn from each point of division to F; and that the arcs DC and AB are transversal to the arcs FD, FP, FB, FQ, they are themselves divided harmonically in K, E and L, E. Similarly AD and BC are harmonically divided in E, F and H, F. So, likewise are GH and KL divided harmonically, as stated in the enunciation.

It is obvious that in this sense of the term "harmonical division of an arc" we may substitute for any arc its supplement: and thus shew the relation to subsist, *mutatis mutandis*, amongst all the parts of the complete quadrilateral. The inquiry is now reduced to too simple a state to need any further detail in a work like this.

Scholium —A great number of properties—in fact all which have any analogy to the complete plane quadrilateral—will follow at once from the preceding investigation. For instance Brianchon's *fasceau harmonique*—the properties of inscribed and circumscribed quadrilaterals in reference to a spherical conic section—Pascal's mystic hexagram —and indeed, all that relates to Poles and Polars *on the sphere*. As far, in fact, as is yet known, this method is the only one by which spherical properties can be completely investigated; and perhaps the transversal on the sphere may ultimately prove to be the perfect substitute for the similar triangles in plano, when we enter into analogous spherical researches. This, however, is only to be understood in reference to geometrical methods in plano and not to the coordinate methods of investigation. On the sphere, the coordinate methods are as immediately applicable as in plano: but this topic is already in good hands—in the hands of the Editors of the "Mathematician," under the form of "poles and polars" and the "collinear theory."

I may add that this proposition was sent to Professor Gill of New York, with whom I was in correspondence on other subjects, several years ago; and that (whether from his oversight or mine, I am not able to say) it was proposed in an imperfect form in his "Mathematical Miscellany" as an exercise. It was pronounced by the American geometers, most truly, that in the form in which it was proposed, no such property existed; and the discontinuance of that valuable periodical prevented my correction of it

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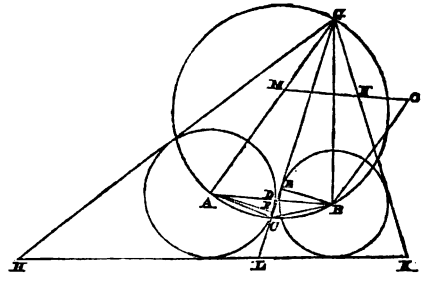
through that medium. I, however, in my edition of Hutton's Course enunciated the theorem in reference to the figure here employed (p. 83); and by the kindness of the editors of the Mathematician have now been enabled to publish the demonstration.

CVII. *W. Burns, LLD., Rochester.*

Given the vertical angle of a triangle, the ratio of the radii of the circles inscribed in the two triangles formed by the line bisecting the vertical angle and meeting the base, and the distance between the centres of these circles, to construct the triangle.

[First Solution. *Messrs. W. H. Levy, Shalbourn; William Mortimer, Seedley, near Manchester; Nicholas Smyth, Galway; and Dr. Burns, the proposer.*]

On AB, the given distance between the centres of the inscribed circles, describe a segment of a circle containing an angle equal to half the given vertical angle; divide AB in D in the given ratio of the radii of the inscribed circles; through D and the middle point of the arc AB draw a line meeting the circumference in G, and draw AF, BE perpendicular to DG. From A and B with the radii AF and BE respectively describe circles touching DG; then from G draw the tangents GH and GK, and also HK, a common tangent to both circles, meeting the former in H and K; then GHK is the triangle required.



For draw AG and BG; then the angle AGD is equal to the angle BGD, since they stand on equal circumferences, by construction; and since angle AGD is equal to angle AGH, and BGD to BGK, the whole angle HGK is evidently double of the angle AGB, and therefore equal to the given angle. Again, by similar triangles ADF, and BDE, AD : BD :: AF : BE; therefore the radii of the inscribed circles are in the given ratio, and AB is the given distance between the centres.

Scholium. By drawing the other common tangent to the two inscribed circles, the preceding construction and demonstration would then apply to the problem :—

Given the vertical angle, the ratio of the radii of the circles described without the triangle, each touching the base, a side, and the line bisecting the vertical angle; and the distance between the centres of these circles, to construct the triangle.

[Second Solution. *Messrs. Hugh Godfray, St. Mark's Road, Jersey; and David Maver, Bon Accord School, Aberdeen.*]

Draw any two lines GA and GB forming an angle equal to half the given angle. On these take two distances GM and GN proportional to the two radii, and join MN, on which produced, if necessary, set off MO equal to the given distance of the centres. Draw OB parallel to GA, meeting GB in B; and also BA parallel to MO. Bisect the angle AGB by the line GL, and with centres A and B describe circles to touch GL. Draw the tangents GH and GK and the double tangent HK; then HGK shall be the triangle required.

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For draw the radii AF and BE to the points of contact of the circles with GL; then the similar triangles GAF and GBE give

$$AF : BE :: GA : GB :: GM : GN;$$

therefore AF has to BE the given ratio; also AB is equal to MO, the given distance, and angle HGK is equal to twice the angle AGB, and therefore equal to the given angle.

[THIRD SOLUTION. *Mr. Joseph Paul, Shooter's Hill.*]

Since the geometrical construction of Exercise 108 is included in the solution of the problem of which this is a particular case, namely, that in which the vertical angle is divided into *any* two parts; I shall give a solution of the more general problem.

Let the given distance AB of the centres be divided in D in the given ratio of the radii, and let the angles CAB and CBA be the halves of the given parts of the vertical angle; through A, C and B describe the circle ABG; join CD, and produce it to meet the circumference in G; draw the perpendiculars AF and BE, with which as radii describe circles; the tangents GH and GK, and the common tangent (see BLAND'S Geo. Pr., sect. 2nd. prob. 7,) HK of those circles shall be the sides of the triangle.

Join GA and GB: then the angles AGC and BGC are respectively equal to ABC and BAC, being in the same segment; therefore their doubles, namely, HGL and KGL, are the given parts of the given angle; it is also manifest that AF : BE :: AD : BD, the given ratio of the radii.

Analytical solutions of this exercise were received from H. W. and E. W., of King's College, London.

CVIII. *Mr. John Riddle, Nautical School, Greenwich.*

An observer a short distance inland, notes the time when a steamer is just passing a land-mark on the beach, and m minutes afterwards the angle subtended by the vessel and the mark is α, and again in n minutes their angular distance is $\alpha+\beta$. The bearing of the land-mark being known, required the course of the ship?

[FIRST SOLUTION. *Messrs. A. R. Clarke, Gentleman Cadet, Royal Military Academy; and John Riddle, the proposer.*]

(See fig. to the solution of exercise 107.)

Let G be the place of the observer, M that of the land mark, and A, D, B the three successive positions of the ship; then angle $AGD = \alpha$, angle $DGB = \beta$, and the rate of the ship being taken as unity, m and n will measure the distances run in m and n minutes respectively, viz. AD, DB.

Now the bearing of M or of A being given, it is evident that if the angle GAB were found, the direction of AB, the course of the ship, would be known. Let angle $GAB = \theta$, then angle $GDB = \alpha+\theta$, and we have

$$(m+n)\sin\theta = BG\sin(\alpha+\beta) \quad \ldots\ldots(1),$$

and

$$n\sin(\alpha+\theta) = BG\sin\beta \quad \ldots\ldots(2).$$

Dividing (2) by (1) gives

$$\frac{n}{m+n}\cdot\frac{\sin(\alpha+\theta)}{\sin\theta} = \frac{\sin\beta}{\sin(\alpha+\beta)},$$

$$\text{or } \frac{n}{m+n}\cdot\frac{\sin\alpha\cos\theta+\cos\alpha\sin\theta}{\sin\theta} = \frac{\sin\beta}{\sin(\alpha+\beta)};$$

$$\therefore \sin\alpha\cot\theta+\cos\alpha = \frac{m+n}{n}\cdot\frac{\sin\beta}{\sin(\alpha+\beta)},$$

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$$\text{or} \quad \cot\theta + \cot a = \frac{m+n}{n} \cdot \frac{\sin\beta}{\sin a \sin(a+\beta)} = \cot\delta, \text{ suppose.}$$

Hence $\cot\theta = \cot\delta - \cot a = \dfrac{\sin(a-\delta)}{\sin a \sin\delta}$, from which the course of the ship may be found.

[SECOND SOLUTION. *Mr. William Mortimer, Seedley; and similarly by Messrs. Joseph Paul, Shooter's Hill; and Dr. Burns, Rochester.*]

Let A, D, B be the three positions of the ship, M the land-mark, and G the place of the observer. Let v = velocity of ship, and angle $GAB = \theta$; then

$$\frac{GD}{\sin\theta} = \frac{mv}{\sin a}, \text{ and } \frac{GD}{\sin(a+\beta+\theta)} = \frac{nv}{\sin\beta};$$

$$\therefore \quad \frac{m\sin\beta}{n\sin a} = \frac{\sin(a+\beta+\theta)}{\sin\theta} = \sin(a+\beta)\cot\theta + \cos(a+\beta);$$

$$\therefore \quad \cot\theta = \frac{m\sin\beta}{n\sin a \sin(a+\beta)} - \cot(a+\beta).$$

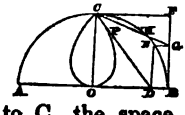
Solutions were received from Messrs. H. Godfray, Jersey; G. F. Lacey, Woolwich Common; N. Smyth, Galway; and H. W. and E. W., King's College.

CIX. *Mr. R. Harley, Northwich, Cheshire.*

ACB is a semicircle whose diameter is AB and vertex C: at any point D in AB draw a perpendicular DE meeting the semicircle in E and draw CD. On CD take CP such that $CP.CD = DE^2$, and find the quadrature of the curve which is the locus of the point P, without the aid of the differential calculus.

[SOLUTION. *W. Burns, L. L. D., St. Margaret's Bank, Rochester; and similarly by Mr. Harley, the proposer.*]

Draw BF and CF touching the circle at B and C, and EG parallel to AB; join CG cutting the circle in H; then since



$$CG.GH = GB^2 = DE^2 = DC.CP;$$

therefore

$$CD : CG :: GH : CP;$$

and as this relation holds true whilst E moves from B to C, the space described by CD is to the space described by CG, as the space described by GH to the space described by CP. Now whilst E moves from B to C, CD describes the half of the square OF, and CG describes the other half of the same square OF; hence GH and CP will also describe equal spaces; but GH describes the space CFBEC; hence the space described by CP is equal to the excess of the square OF above the quadrantal area OBEC. Hence if $OC = a$, the area of the curve described by P, on both sides of the axis OC, is $= 2a^2 - \frac{1}{2}\pi a^2 = \dfrac{a^2}{2}\left(4 - \pi\right)$.

Ingenious solutions were sent by Messrs. Joseph Paul, Shooter's Hill, and Nicholas Smyth, Galway.

CX. *Mr. J. W. Elliott, Greatham, Stockton-on-Tees.*

A straight uniform beam is placed upon two rough planes, whose coefficients of friction are μ and μ', and inclinations to the horizon a and β; to find the limiting value of the angle of inclination of the beam to the horizon at which it will rest, and the relation between the weight of the beam and each of the pressures.

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